Road Coloring

2017 – 2018
Student Workbook

THE ALGEBRA PROJECT

Road Coloring

Table of Contents

Name: ______________________________ Teacher: ____________________________ Date: _____________

Section 1 - Building Cities

Activity 1.1 – CLASSWORK – Discussion: Properties of a City

As we said before, each lesson in this class begins with an experience. There will be puzzles, races, and games. After each experience, you'll talk about what happened. From our conversations, we'll begin to learn math.

Our first experience is a puzzle. Your first job is to build a city. Your city will be a collection of buildings with one-way roads connecting the buildings together. The cities you'll build will have special features, so that you can try to solve the puzzle. Here are the features for the cities:

- **Feature A** - Each building must have exactly two roads leading away from it.
- **Feature B** - If you start in any building, there has to be a sequence of roads that can get you to every building in your city, including back to where you started.

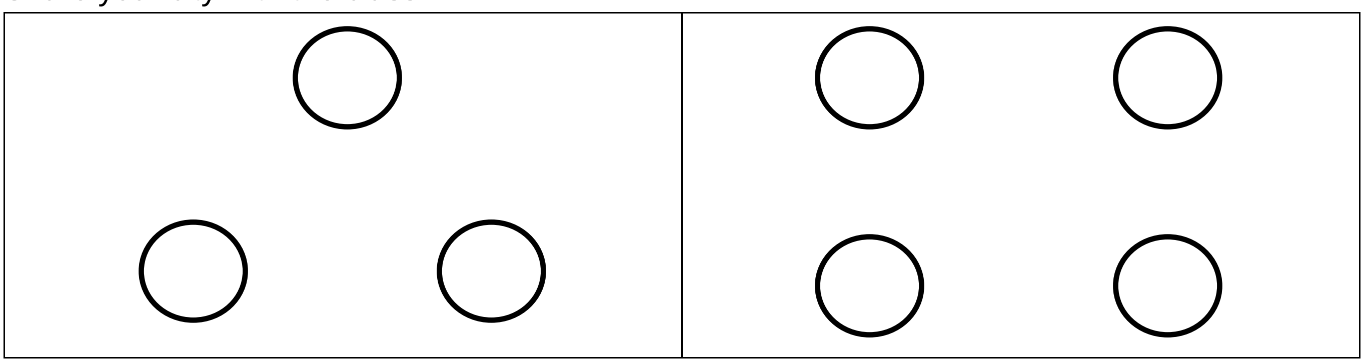
We'll refer to these specific rules by name, but here are some additional important rules:

- Notice that the first feature only talks about how many roads leave from a building. It doesn't say anything about how many roads go to a building.
- There can be one or more roads going to a building.
- A road can leave from a building and come right back to the same building. This kind of road is called a loop.
- Both roads from a building can go to the same place.
- Roads can't connect to other roads.

Activity 1.2 – TEAM WORK – Build a City

In this activity, teams will construct cities with the materials provided. Remember to make sure that the cities satisfy Feature A and B listed above.

- Build a city with 3 buildings in it and draw it below.
- Share your city with the class.
- Build another city with 4 buildings in it and draw it below.
- Share your city with the class.

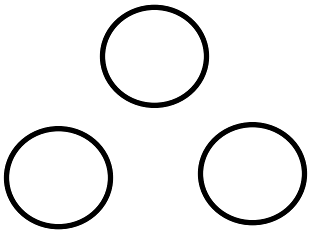
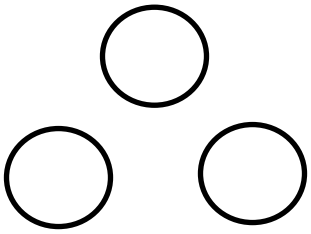
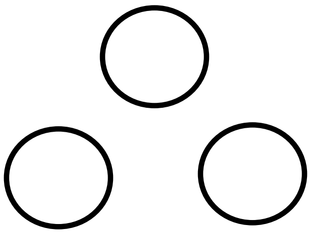
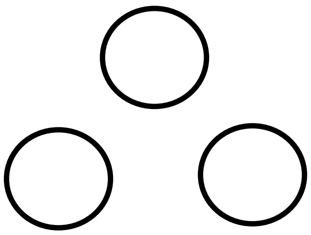
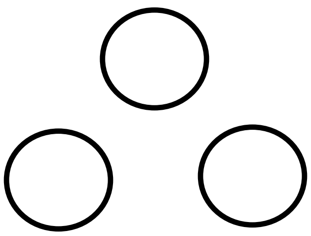
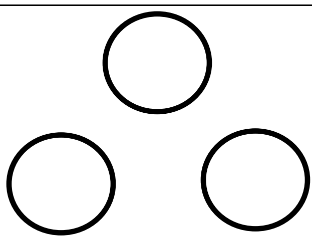


Team members: ____ Team Name: _________________

Activity 1.3 – TEAM WORK – City Diagrams Practice

For each of the cities that teams built, draw them out and explain whether it is a correct city or not.

Table 1.3

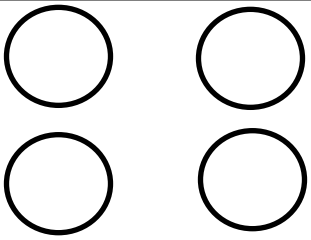
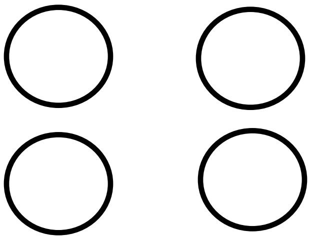
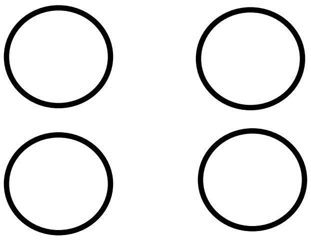
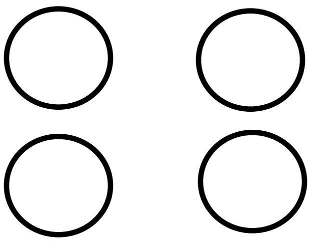
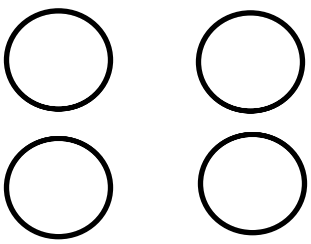
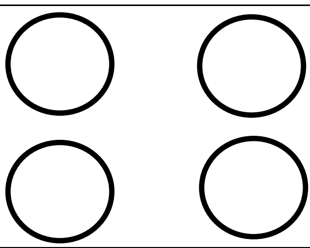
City Diagram	Is it a city?	If **No**, why not?
	**YES** **NO**	
	**YES** **NO**	
	**YES** **NO**	
	**YES** **NO**	
	**YES** **NO**	
	**YES** **NO**	

Team members: ____ Team Name: __________________

Activity 1.4 – TEAM WORK – City Diagrams Practice

For each of the cities that teams built, draw them out and explain whether it is a correct city or not.

Table 1.4

City Diagram	Is it a city?	If **No**, why not?
	**YES** **NO**	
	**YES** **NO**	
	**YES** **NO**	
	**YES** **NO**	
	**YES** **NO**	
	**YES** **NO**	

Team members: ____ *Team Name:* _________________

Section 2 – Naming the Roads

Activity 2.1 – TEAM WORK – Coloring Your Roads

Now that we know how to build a city, the next step is to name the roads so we can give directions to people who want to travel around the city. We'll use colors (red and blue) to name the roads. We only need two colors, since each building has exactly two roads leading from it.

Once we name the roads, we can take a tour of city, so we'll call these cities **touring-cities**. To turn a city into a touring-city every road needs to have a color assigned to it, either red or blue.

A city becomes a touring-city if every building has EXACTLY one red road and one blue road leading from it.

Sometimes people think that there is only one correct answer in math. In some cases, this is true. In this case, there will be many correct ways to turn a city into a touring-city.

Turn each of the cities your team designed in **Activity 1.2** into touring-cities by correctly assigning colors to each of the roads:

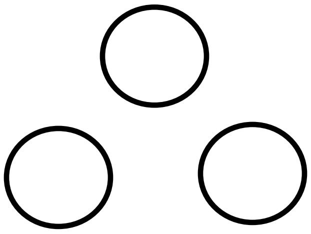
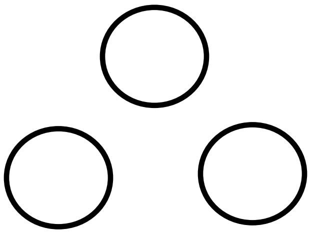
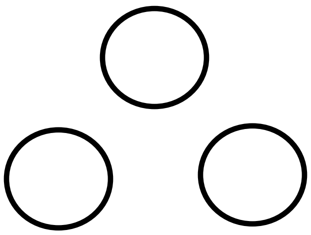
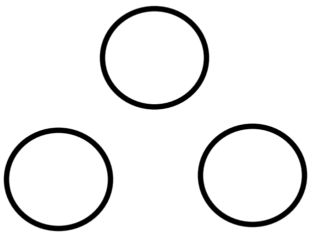
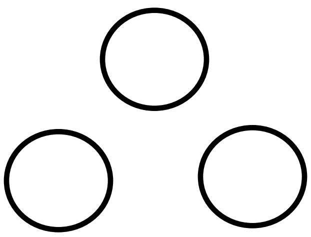
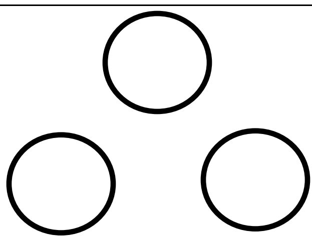
- Do this for your three-building city.
- Each team should report to the class the **touring-cities** they designed.
- You may need to rebuild your city if it has more than one road leading to it and one road leading away from it.
- Repeat for your four-building city.

Name: ________________________________ Teacher: ______________________________ Date: ______________

Activity 2.2 – INDIVIDUAL WORK – Touring Cities

For each of the touring-cities that teams built, draw them out and explain whether it is a correct city.

Table 2.2

City Diagram	Is it a touring-city?	If **No**, explain for each feature.
	**YES** **NO**	Feature A – Feature B –
	**YES** **NO**	Feature A – Feature B –
	**YES** **NO**	Feature A – Feature B –
	**YES** **NO**	Feature A – Feature B –
	**YES** **NO**	Feature A – Feature B –
	**YES** **NO**	Feature A – Feature B –

Activity 2.3 – INDIVIDUAL WORK – Touring Cities

For each of the touring-cities that teams built, draw them out and explain whether it is a correct city.

Table 2.3

City Diagram	Is it a touring-city?	If **No**, explain for each feature.
	YES **NO**	Feature A – Feature B –
	YES **NO**	Feature A – Feature B –
	YES **NO**	Feature A – Feature B –
	YES **NO**	Feature A – Feature B –
	YES **NO**	Feature A – Feature B –
	YES **NO**	Feature A – Feature B –

Section 3 – Solving the Puzzle

Activity 3.1 – CLASSWORK – Discussion: Defining the Puzzle

Now that we have designed touring-cities, there are many different things that we can do. We will begin to try and solve a famous puzzle in this section. At the start of our puzzle there will be one person in each building in our city.

Find a set of directions that gets all the people in the city to the same building, at the same time.

One possible example for a set of directions would be, “First follow the red road, then follow the blue road.”

What this set of directions means is that **all of the people** in the city will:

1. Find the red road that leaves the building they are in, and follow it to the next building. (*If the road is a loop, you will go back to the same building*.)

2. Then find the blue road that leaves that building and follow it to the next building.

Now that we have fully described the puzzle, we need to start keeping track of the attempts to solve the puzzle. To make things easier, we’ll start using a shorthand notation to refer to a set of directions. If we say, “First follow the red road, then follow the blue road,” we’ll just write down “Red-Blue.”

If you can get everyone in the same building at the same time, you have solved the puzzle!

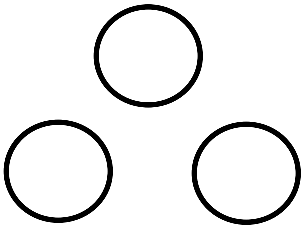
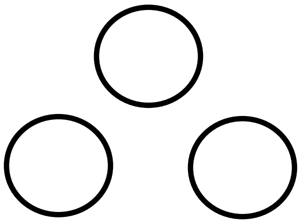
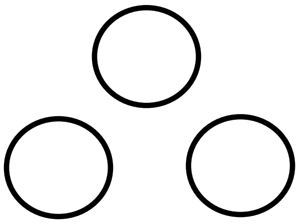
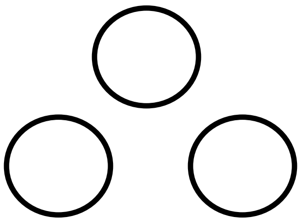
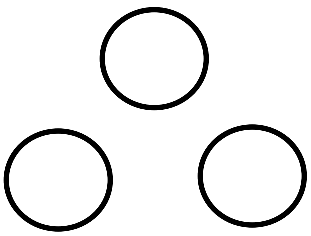
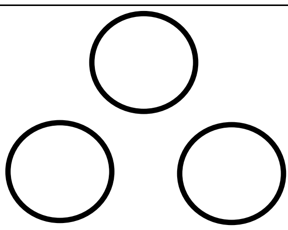
As a class, pick four students to act out one of the correctly drawn touring-cities. You may need to redesign your city with the permission of your teacher.

Name: ______________________________ Teacher: ____________________________ Date: _____________

Activity 3.2a – INDIVIDUAL WORK – Road Coloring Practice

For each of the touring-cities that teams built, try to solve the puzzle. If it's not possible, explain why?

Table 3.2a

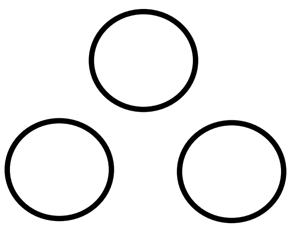
Correctly Colored City Diagram	Was a Solution Found?	If **Yes**, what was the solution? If **No**, why not?
	**YES** **NO**	
	**YES** **NO**	
	**YES** **NO**	
	**YES** **NO**	
	**YES** **NO**	
	**YES** **NO**	

Team members: ____ Team Name: ____________________

Activity 3.2b – TEAM WORK – Discussion Questions

As a team, answer the following questions.

Table 3.2b

A. What is a good name for that special set of directions that gets everyone to the same building?
B. Have you been able to find a solution for every city? Do you think it is always possible to find a solution? Why or why not?
C. Use one of the cities for which you were able to find a solution and then switch the color of the roads leading from Building 1. *(That is, change red to blue and blue to red.)* Will the same set of directions still be a solution for the new coloring? Why or why not? *(You might need to actually build the city and walk it to determine the answer.)*
D. Can you design a city and color the roads correctly so that a single command gets everyone to the same building? (*Hint: You'll need a loop.)* Show your design below and describe why it works. Write a story that describes why you might want to design a good city in this way. 

Name: ______________________________ Teacher: ____________________________ Date: _____________

Activity 3.3a – INDIVIDUAL WORK – Road Coloring Practice

For each of the touring-cities that teams built, try to solve the puzzle. If it's not possible, explain why?

Table 3.3a

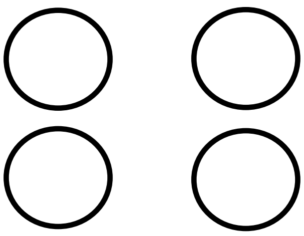
Correctly Colored City Diagram	Was a Solution Found?	If **Yes**, what was the solution? If **No**, why not?
	YES **NO**	
	YES **NO**	
	YES **NO**	
	YES **NO**	
	YES **NO**	
	YES **NO**	

Team members: ____ *Team Name:* _________________

Activity 3.3b – TEAM WORK – Discussion Questions

As a team, answer the following questions.

Table 3.3b

A. Would you change your answer to Question A in Activity 3.2b after doing 3.3a? Why or why not?
B. Would you change your answer to Question B in Activity 3.2b after doing 3.3a? Why or why not?
C. Use one of the cities for which you were able to find a solution and then switch the color of the roads leading from 1 of the buildings and mark the building you chose with a star. *(That is, change red to blue and blue to red.)* Will the same set of directions still be a solution for the new coloring? Why or why not? *(You might need to actually build the city and walk it to determine the answer.)*
D. Can you design a city and color the roads correctly so that a single command gets everyone to the same building? (*Hint: You'll need a loop.)* Show your design below and describe why it works. Write a story that describes why you might want to design a good city in this way. 

Team members: ____ Team Name: _________________

Section 4 – Models and Representations

Activity 4.1 – TEAM WORK – Create a Representation

Next comes one of the most important parts of learning to do math: building models. A model is something that makes the experience **present** to us. Since the experience is present again when we have the model, we say the experience is "**re-present**." In other words, the model is a "**representation**."

It doesn't matter what form you choose. If you like to draw, draw a picture that represents solving the puzzle. If you like music, write a song. Or come up with your own ideas for a way to represent solving the puzzle.

- Each team should decide how they are going to represent solving the puzzle. Create your team's representation (or representations).
- After your team has come up with a representation, you will present it to the class.

Activity 4.2 – CLASS WORK – Class Discussion: The Road Coloring Problem

Each of the models you and your classmates created has properties that reflect aspects of the experience of walking around in your city, attempting to get everyone to the same building. We will discuss the representations which are used by the research mathematicians who study this puzzle.

First of all, it is important to understand that this puzzle, or problem, is currently being studied at universities around the world. This is probably the first example that you have seen of a mathematics problem that is was solved in your lifetime. All of the math problems you have seen in your educational careers were solved hundreds of years ago. This is a problem that was solved in your lifetime.

What problem? For obvious reasons, the puzzle we have been discussing is called the **Road Coloring Problem**.

You may be saying to yourself, "But wait, we've solved the problem a bunch of times already!"

In one sense, you are right. You and your classmates have found solutions for specific cities with three buildings, but to solve the Road Coloring Problem, a mathematician must be able to prove that for every good city there is a way to color its roads so that a set of directions can be found to get everyone to the same building.

They have to show it works for cities with 100 buildings, or 1,000,000 buildings, or 1,000,000,000,000,000 buildings. And they have to prove that it works. (We'll discuss later this important idea of what a mathematician means by proof.)

Models and Representations

A good mathematical representation finds the exact features of a problem that need to be preserved and eliminates everything else. For our cities, the features that are important are the buildings and the roads. This leads to the model called a **directed graph**. For example, we could build the following city:

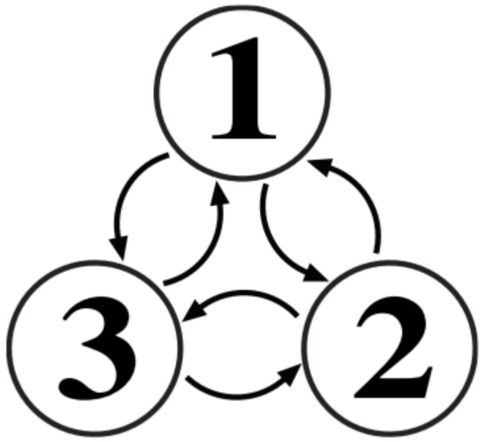


Figure 4.2a – Three building city modeled as a **Directed Graph.**

The numbered circles represent the buildings and the one-way roads are shown as arrows. By representing the problem in this manner, we have concentrated on the key parts of our city. In mathematics, the circles are called **vertices** (one circle is called a **vertex**) and the arrows are called **direct edges**.

Now that we have our directed graph representation of this city, let's begin to run some experiments. Math is like science, we need to perform experiments on our models to completely understand their properties.

The second step of the puzzle involved coloring the roads of our city. Here is one possible way to color roads of the city from Fig. 4.2a.

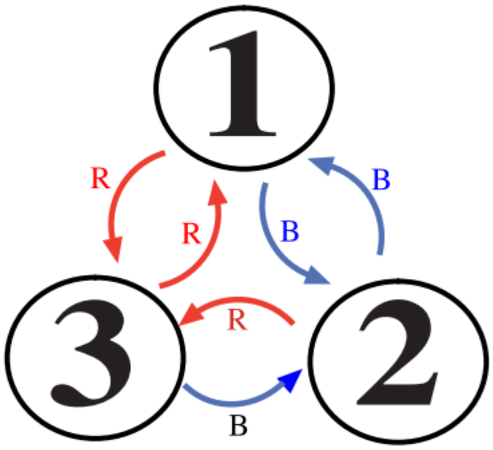


Figure 4.2b - Edge-colored directed graph from Figure 1.

You may have already experimented with this city and this particular way to color the roads. If someone in your class built this city during your experiment, they may have colored it differently, but let's use this as an example. We will refer to the directed graph of a city that is correctly colored as a "city map."

Now we'll carefully observe what happens when the command "Blue" is called out. We'll put names in each of the buildings to represent the people walking around the city.

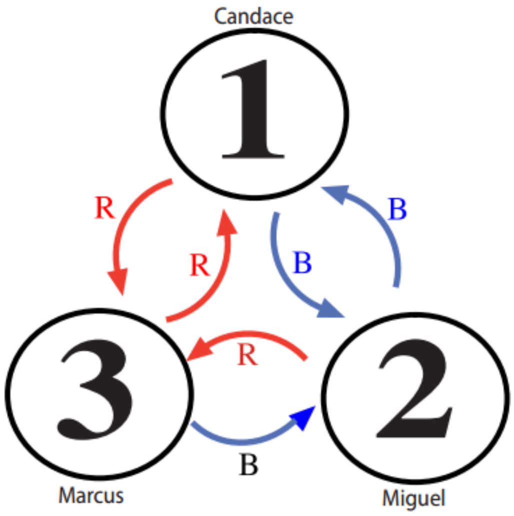


Figure 4.2c - When the command "Blue" is called out, here is what happens to the people in the city.

- *Candice **leaves from** building 1 and **goes to** building 2.*
- *Miguel **leaves from** building 2 and **goes to** building 1.*
- *Marcus **leaves from** building 3 and **goes to** building 2.*

Now if the command "Red" is called out next, we can see the following occur:

- *Candice **leaves from** building 2 and **goes to** building 3.*
- *Miguel **leaves from** building 1 and **goes to** building 3.*
- *Marcus **leaves from** building 2 and **goes to** building 3.*

We've done it! The set of directions "Blue-Red" gets everyone to building 3. Mathematicians refer to a set of directions that solved the puzzle as "**synchronizing**." (When your team answered question 1 in Activity 3.3 on page 9, did they decide on a different name?) The word synchronizing comes from the Greek language and means "same time." ["Syn" means same, like in **synonym**; "chronos" means time like in **chronology**.]

For the worksheets that follow, choose some names of your classmates and place them in the vertices that represent the buildings of the three building cities your classes designed. Then record their movements around the city as the commands are called.

Name: ______________________________ Teacher: ____________________________ Date: _____________

Activity 4.3 – INDIVIDUAL WORK – Solving the Road Coloring Problem

Table 4.3

City Diagram	Command and Directions
1 3 2	First Command _________ leaves building _______ & goes to building ______ _________ leaves building _______ & goes to building ______ _________ leaves building _______ & goes to building ______
1 3 2	Second Command _________ leaves building _______ & goes to building ______ _________ leaves building _______ & goes to building ______ _________ leaves building _______ & goes to building ______
1 3 2	Third Command _________ leaves building _______ & goes to building ______ _________ leaves building _______ & goes to building ______ _________ leaves building _______ & goes to building ______
1 3 2	Fourth Command _________ leaves building _______ & goes to building ______ _________ leaves building _______ & goes to building ______ _________ leaves building _______ & goes to building ______
1 3 2	Fifth Command _________ leaves building _______ & goes to building ______ _________ leaves building _______ & goes to building ______ _________ leaves building _______ & goes to building ______

Activity 4.4 – CLASS WORK – Class Discussion: Arrow Diagrams

Let's continue our analysis of the commands. We'll go back to the city from Fig. 4.2b on page 11.

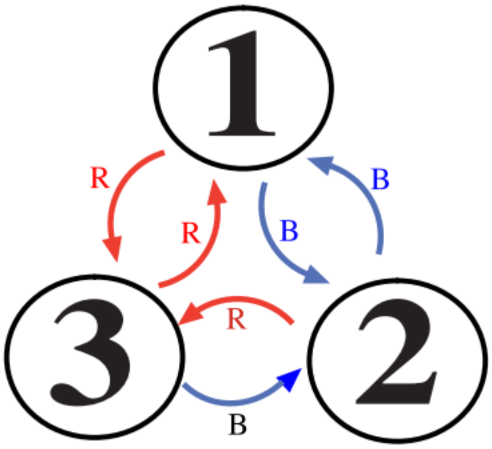


Figure 4.4a – Directed graph representation of a city.

In the course of solving this puzzle, many different people may have occupied the buildings. No matter who was in building 1 when the command "Blue" was called out, that person moved to building 2. The name of the person in building 1 didn't affect what happened when the command was called out. If Marcus was in building 1, and blue was called, Marcus would go to building 2. If Candice was in building 1, the same thing would have happened. Candice would have gone from building 1 to building 2. This is an important idea in Math. Mathematicians try to notice the properties which do not change.

The "function" of an object is what is accomplished by that object. So, what is the "function of the blue roads?" The following representation describes this function.

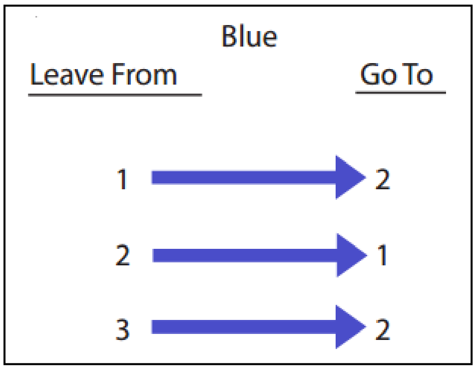


Figure 4.4b – Arrow diagram of blue roads from Figure 4.4a.

While we're at it, we might as well represent the function of the red roads.

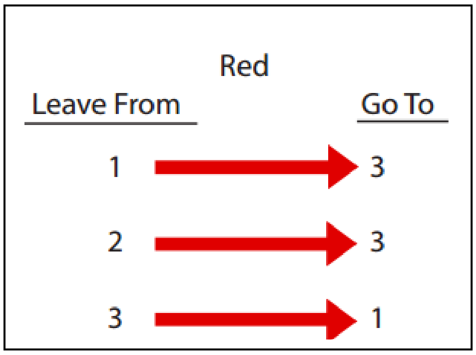


Figure 4.4c - Arrow diagram of red roads from Figure 4.4a.

Now it is time for some careful observation:

- Since there is a red road that **leaves from** every building each of the numbers (1, 2, 3) are in the **leaves from** column. The same is true for the blue road.
- Not every building has a blue road which **goes to** it. In fact, only buildings 1 and 2 have blue roads which go to them. The number 3 is missing from the **go to** column

Sometimes it is as important to represent what doesn't happen as it is to represent what does happen. We should represent the fact that no blue road goes to building 3. Here is a representation which does the trick.

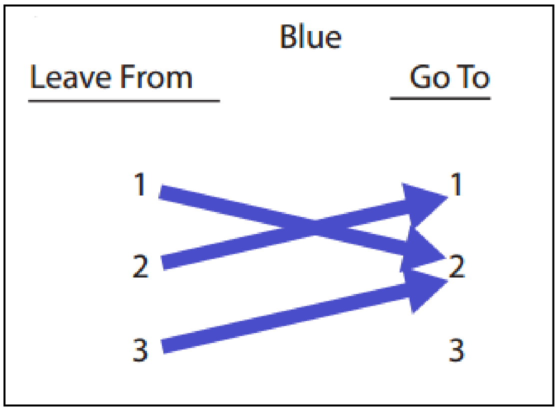


Figure 4.4d - Arrow diagram of blue roads from the city in Figure 4.4a..

Now we have all of the buildings in the "Go to" column. The fact that no blue road goes to building 3 is represented by not having an arrow lead to building 3. We don't need to label the columns anymore since the arrows themselves tell the story. The arrows "leave from" the buildings on the left, and "go to" the buildings on the right.

The function of the red roads can be represented in this way.

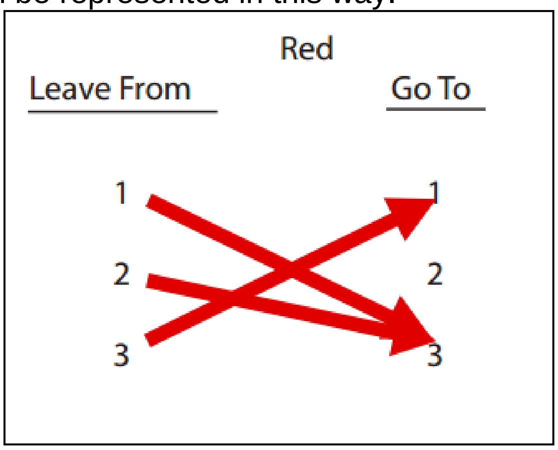


Figure 4.4e - Arrow diagram of red roads from the city in Figure 4.4a..

You can immediately see from this representation (without looking at the directed graph!) which building does not have a red road that goes to it. This new representation is important enough for us to give it a name. We'll call this representation an **arrow diagram**.

Remember, the <u>arrow heads are important</u>, since these are one-way roads. The arrows point toward the "go to" building numbers. DO NOT JUST DRAW LINES!

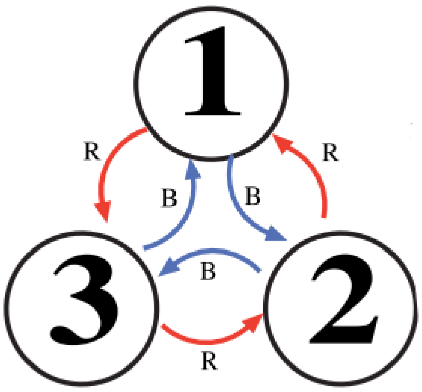
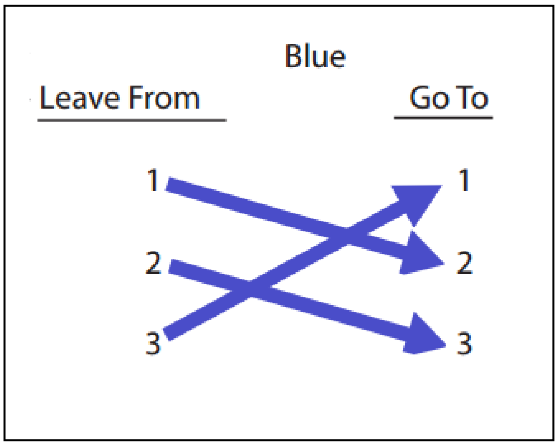
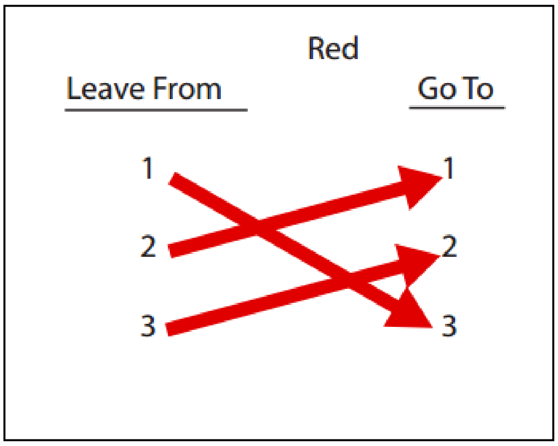


Figure 4.4f

Here is another way to color the city, and the arrow diagrams for the functions of the red and blue roads.





Figures 4.4g and 4.4h – Arrow diagrams for the coloring of the city in Figure 4.4f.

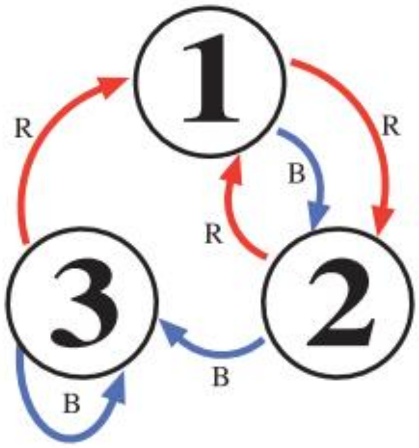
Name: ______________________________ Teacher: ____________________________ Date: _____________

Activity 4.5 – INDIVIDUAL WORK – Arrow Diagrams from Directed Graphs

For each city below, draw the arrow diagrams for the functions of the blue and red roads.

Blue

Leave From	Go To
1	1
2	2
3	3

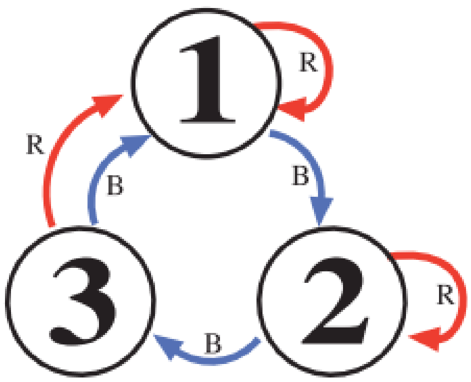


Red

Leave From	Go To
1	1
2	2
3	3

Blue

Leave From	Go To
1	1
2	2
3	3



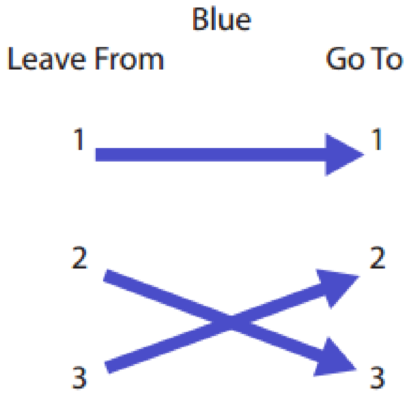
Red

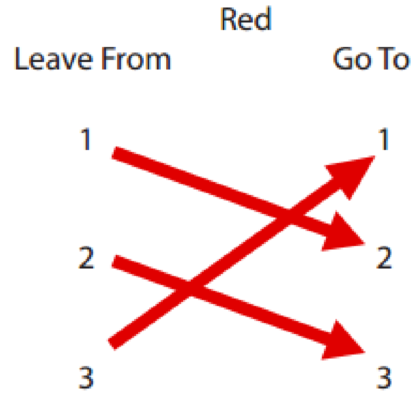
Leave From	Go To
1	1
2	2
3	3

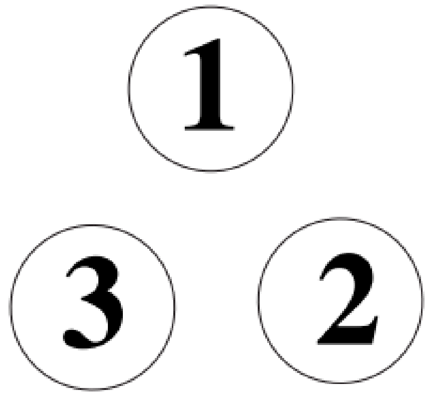
Name: ______________________________ Teacher: ____________________________ Date: _____________

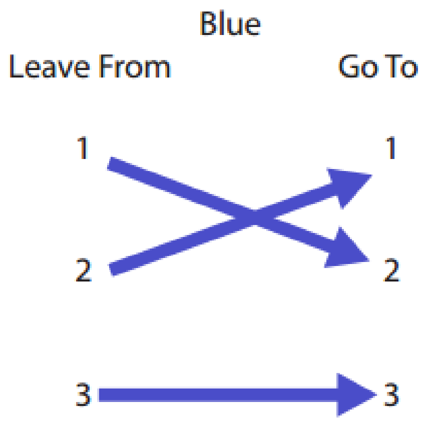
Activity 4.6 – INDIVIDUAL WORK – Directed Graphs from Arrow Diagrams

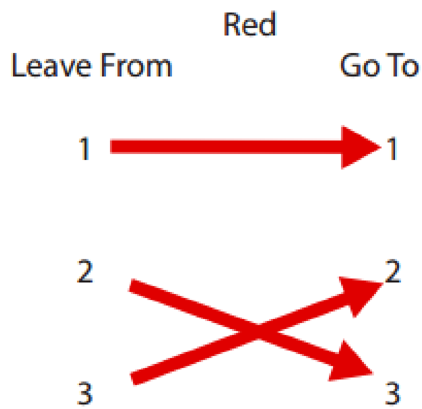
For each city below, draw the directed graph for the functions of the blue and red roads.

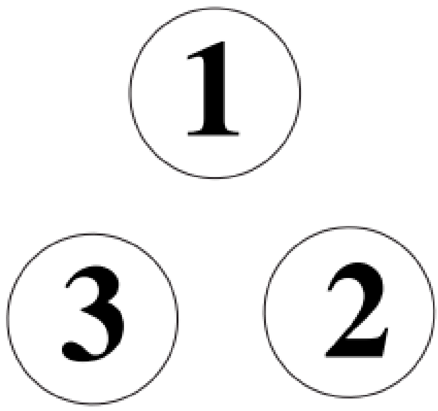












Activity 4.7 – CLASS WORK – Class Discussion: Road Subgraphs

One easy representation that gives us the same information as an arrow diagram is the part of the directed graph we get by just concentrating on either the red or blue roads. Let's look at the city below.

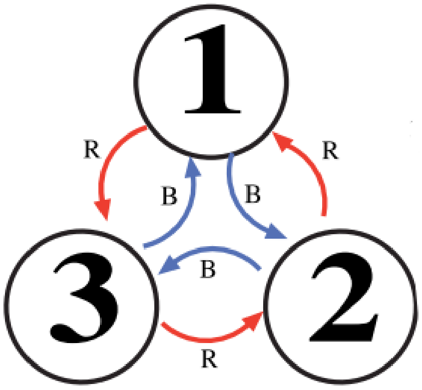
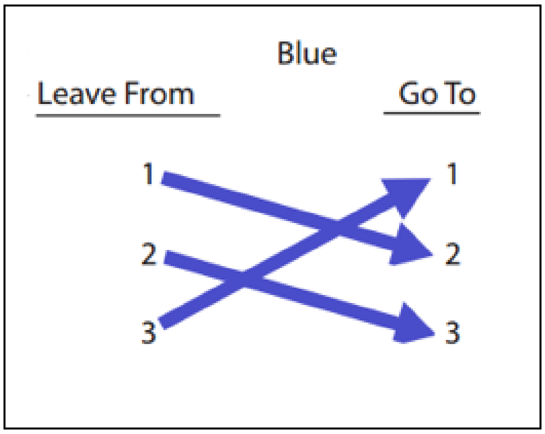
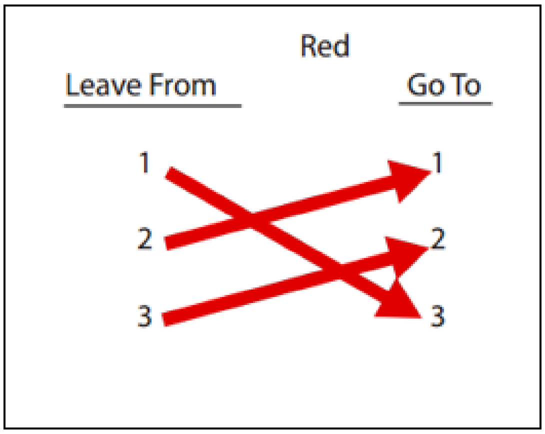


Figure 4.7a

Here are the arrow diagrams for the red and blue roads.





Figures 4.7b and 4.7c – Arrow diagrams for the city depicted in Figure 4.7a

What would our city look like if we "erased" all the red roads? It would look like this.

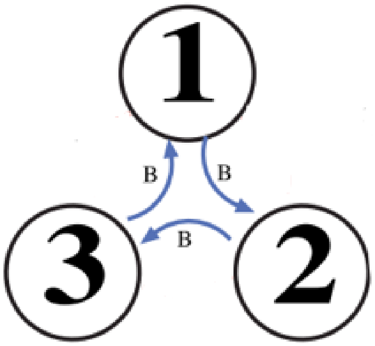


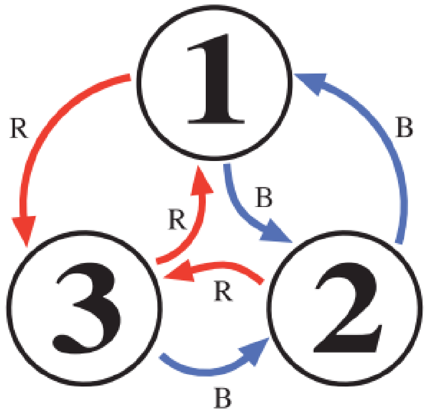
Figure 4.7d – Blue road subgraph from Figure 4.7a

In mathematics, we use the prefix "sub" to mean something which is a part of something else. Since the blue roads represent only a part of the directed graph of the city, we will call the figure above the **subgraph** for the blue roads.

Name: ________________________________ Teacher: ______________________________ Date: ______________

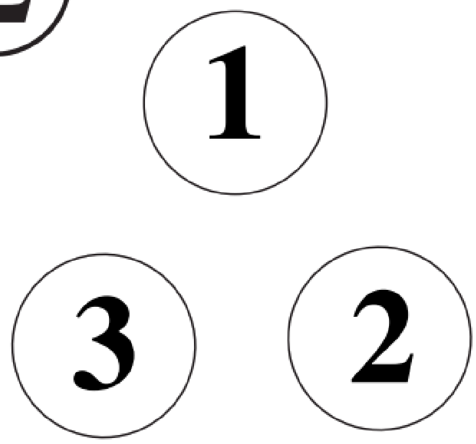
Activity 4.8 – INDIVIDUAL WORK – Directed Graphs from Arrow Diagrams

For the city below, draw the arrow diagram and subgraph for the functions of the blue and red roads.



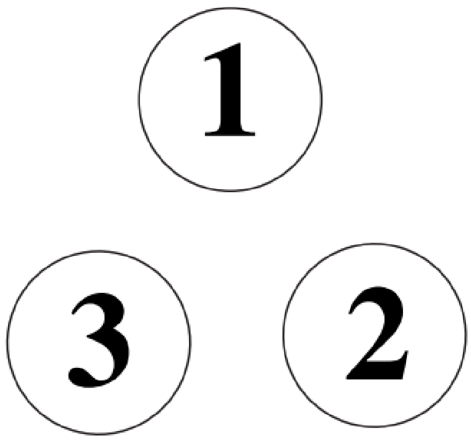
Red

Leave From	Go To
1	1
2	2
3	3



Blue

Leave From	Go To
1	1
2	2
3	3



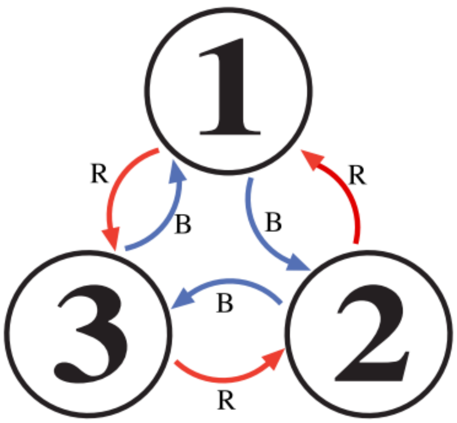
Section 5 – Using Representations and the Cartesian Plane

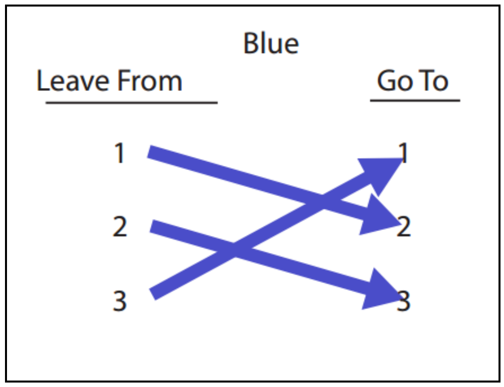
Activity 5.1 – CLASS WORK – Class Discussion: Arrow Diagrams and Ordered Pairs

We'll continue our analysis of the problem by finding another representation for the functions of the red and blue roads. Just as each of the representations you and your classmates created emphasized different aspects of the experience, the representations mathematicians use also have different characteristics.

An arrow diagram has different parts. There are numbers for the "leave from" buildings and the "go to" buildings. There are also arrows connecting the building numbers and the name of the command that you say to move the people. Let's focus on one arrow of an arrow diagram.

Remember, each arrow in an arrow diagram has a "leave from" part and a "go to" part. Look at the arrow in the "Blue" arrow diagram that leaves from building 2 and goes to building 3. If you were in building 2 when someone called out "Blue," you would start in building 2 and finish in building 3. In other words, your movement would have an order to it. First you are in building 2 and second you are in building 3.





Figures 5.1a and 5.1b – A city diagram and its arrow diagram for the blue roads.

An easy way to represent this ordered movement is by putting the numbers in an **ordered pair** (2,3). The number 2 is called the first **coordinate** and the number 3 is called the second **coordinate**. (The word co-ordinate comes from Latin where "co" means with. In other words, a coordinate is "something we order with.") We can represent each of the arrows of the blue arrow diagram as an ordered pair. If we put all of the ordered pairs together in a collection, we would get something like this:

Blue

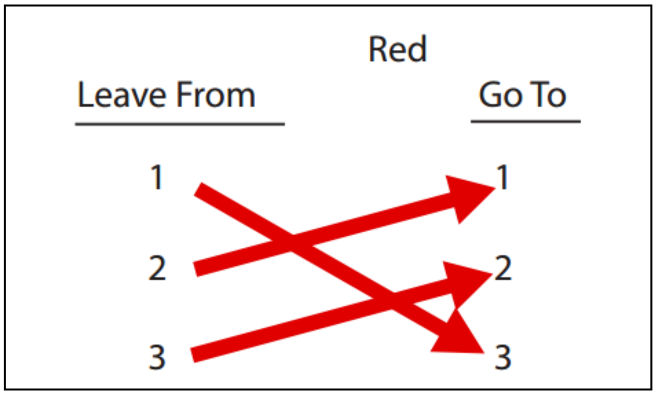
(1,2)

(2,3)

(3,1)

Figure 5.1c – Ordered pairs representing the city in Figure 4.9a

We say that the function of the blue roads is represented by ordered pairs.
Here are the ordered pairs for the function of the red roads.



Red

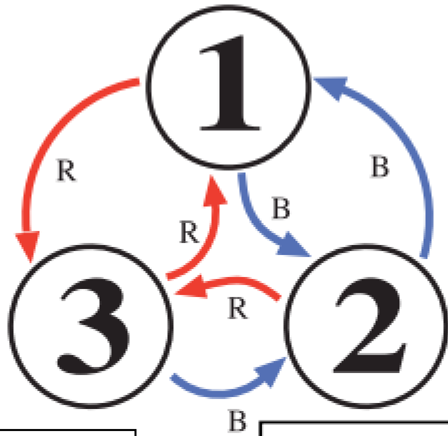
(1,3)

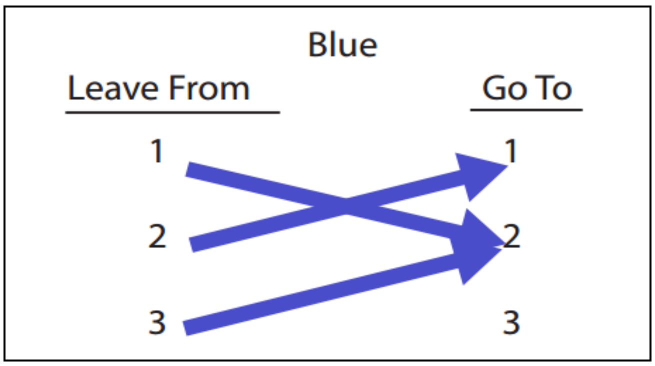
(2,1)

(3,2)

Figures 5.1d and 5.1e – Red arrow diagram and ordered pair representations of the city in Figure 4.9a.

Another example of a familiar coloring. Let’s see the arrow diagrams and ordered pairs.





Blue

(1,2)

(2,1)

(3,2)

Red

(1,3)

(2,3)

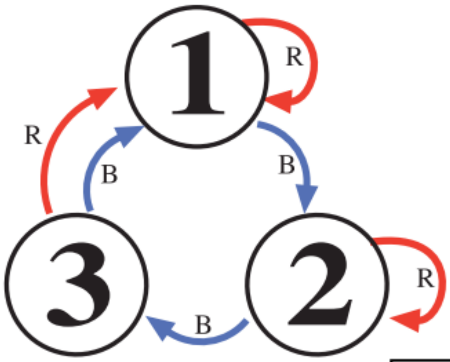
(3,1)

Figure 5.1f – Directed graph, arrow diagram, and ordered pair representations of a city.

Name: ________________________________ Teacher: ______________________________ Date: ______________

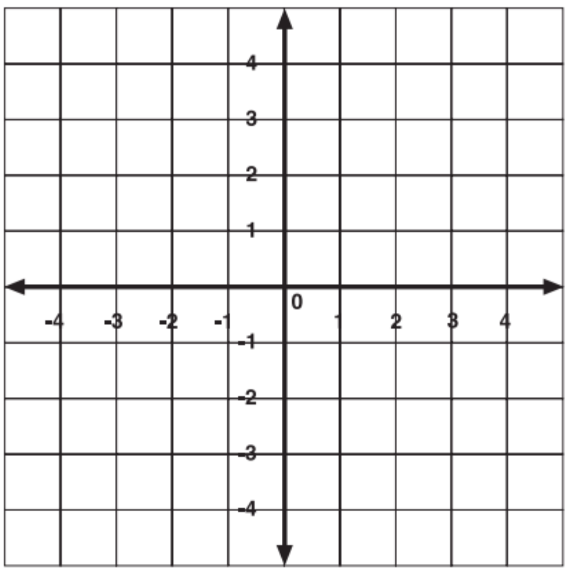
Activity 5.2 – INDIVIDUAL WORK – Arrow Diagrams and Ordered Pairs Practice

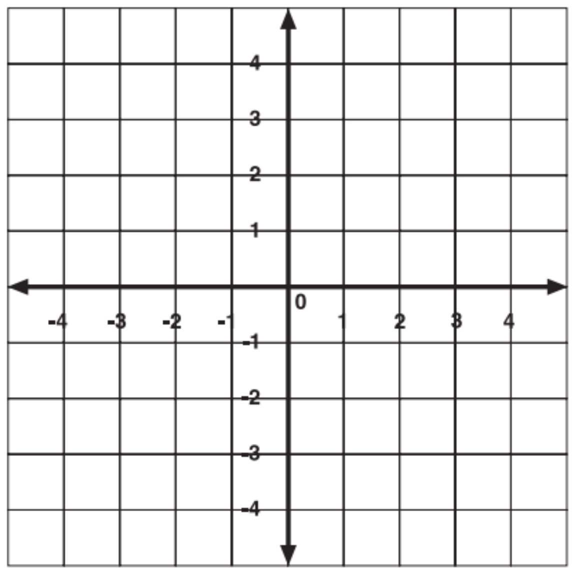
For the cities below, draw the arrow diagram and graph the ordered pairs for the functions of the blue and red roads.

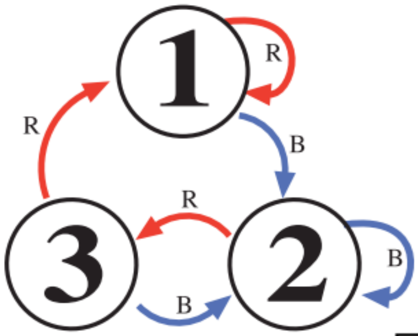


Blue		Ordered pairs for B
Leave From	Go To	
1	1	
2	2	
3	3	

Red		Ordered pairs for R
Leave From	Go To	
1	1	
2	2	
3	3	

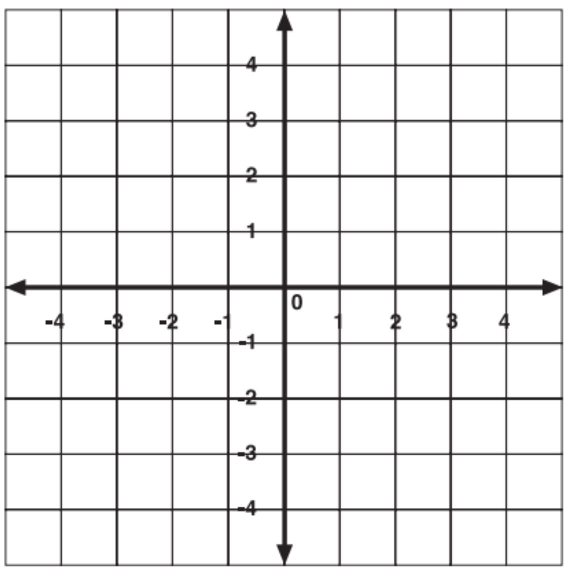


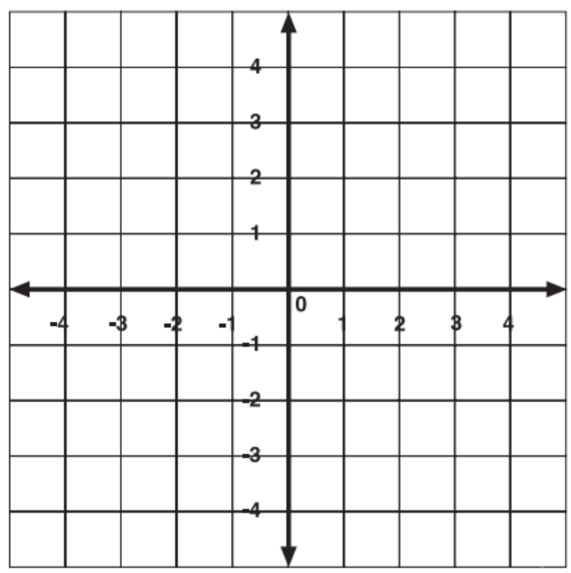




Blue		Ordered pairs for B
Leave From	Go To	
1	1	
2	2	
3	3	

Red		Ordered pairs for R
Leave From	Go To	
1	1	
2	2	
3	3	





Activity 5.3 – CLASS WORK – Class Discussion: Representation Review

We now have four representations for the road functions. We started with arrow diagrams. Then each arrow in an arrow diagram became an ordered pair. The whole arrow diagram became a road subgraph. We also drew some coordinate axes and graphed the ordered pairs as points in the plane.

We want to begin to identify some properties that all of our function representations have, but first we need some new math terms. What terms have we used for the "leave from" and "go to" buildings in our different representations?

Table 5.3

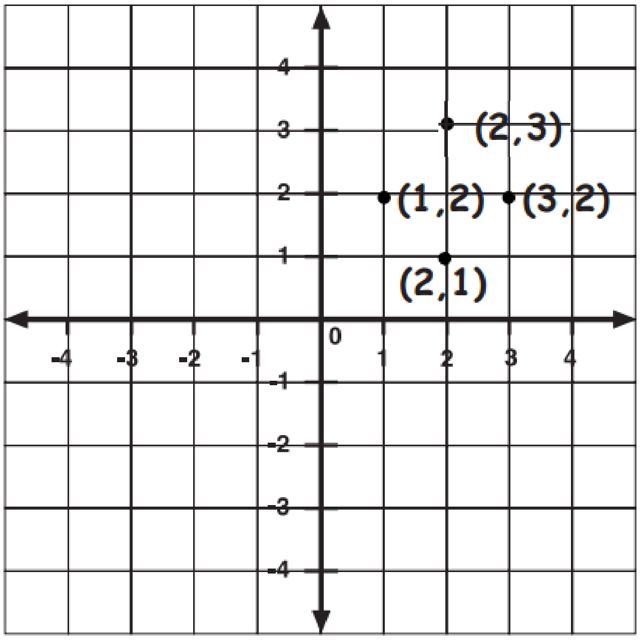
Representation	Mathematical Term for "leave from" building	Mathematical Term for "go to" building
Arrow diagram	"leave from" building number	"go to" building number
Road subgraph	"leave from" building circle	"go to" building circle
Ordered pair	1st coordinate	2nd coordinate
Graph of points	x-coordinate	y-coordinate

Team members: ____ Team Name: _________________

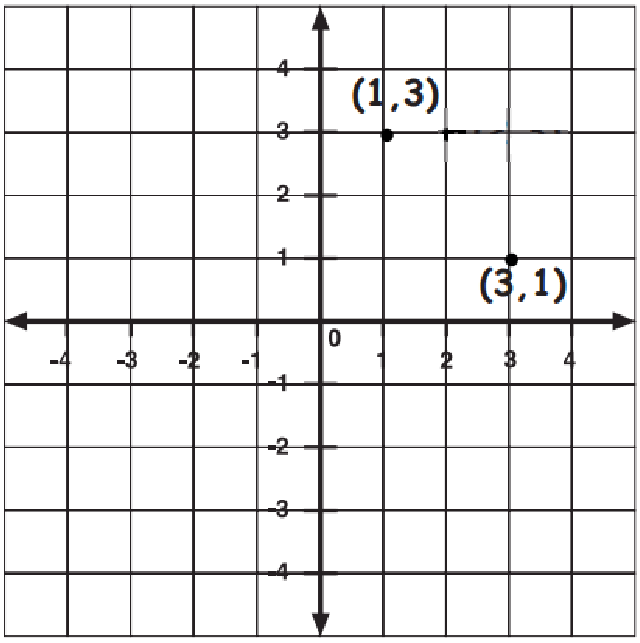
Activity 5.4 – TEAM WORK – Understanding Multiple Representations

By this time, if you are given a graph, you should be able to produce the ordered pairs and the arrow diagram that corresponded to it. As a team, use the coordinate planes to make the arrow diagrams, graph, and answer the questions that follow. Then, try to solve the puzzle (get everyone to the same building with one set of direction)

BLUE

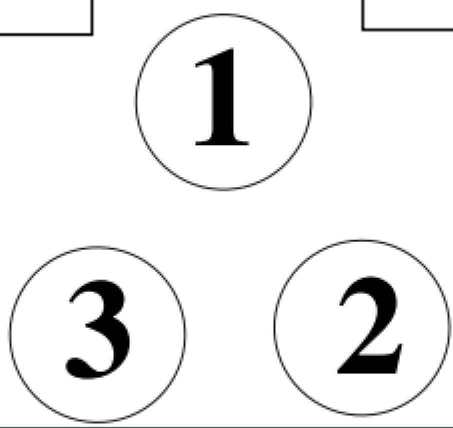


RED



Blue		Ordered pairs for B
Leave From	Go To	
1	1	
2	2	
3	3	

Red		Ordered pairs for R
Leave From	Go To	
1	1	
2	2	
3	3	



Team members: ____ Team Name: ________________

A. As a team, talk about what happened when your team tried to solve the puzzle. What happens to the person in building two when the command “red” is called?
B. How can you tell from the red road arrow diagram that there is something wrong with this coloring of the city roads?
C. How can you tell from the red ordered pairs that there is something wrong with this coloring of the city roads?
D. How can you tell from the red coordinate graph that there is something wrong with this coloring of the city roads?

Team members: ____ Team Name: _________________

E. What happens to the person in building two when the command blue is called?

F. How can you tell from the blue arrow diagram that there is something wrong with this coloring of the city roads?

G. How can you tell from the blue ordered pairs that there is something wrong with this coloring of the city roads?

H. How can you tell from the blue coordinate graph that there is something wrong with this coloring of the city roads?

Section 6 – Functions

Activity 6.1 – CLASS WORK – Class Discussion: Functions

Our last example showed how things can go wrong in a city, if the roads are not properly colored. The ideas in this example are very important. Let's focus on them again.

When "Blue" was called, the person in building 2 didn't know what to do because there were **two blue roads** leading away from building 2. You don't want confusion when you design a city, so we want to eliminate this possibility.

When "Red" was called, the person in building 2 didn't know what to do because there were **zero red roads** leading away from building 2. This is another flaw in the design of a city we want to eliminate.

In fact, this is why when we described how to color the roads of a city in Section 3, we said every building in the city must have exactly **one red road** and **one blue road** leading away from it.

Because of this rule for coloring the roads of a city, we forced the blue and red roads to form a particular type of relationship between the "leave from" buildings and the "go to" buildings. This type of relationship is so important that mathematicians give it the name, **function.**

The "Leave From" building numbers are called the **domain** of the function and "Go To" building numbers are called the **range** of the function.

From now on in your educational careers, virtually everything you encounter in mathematics will be a function of some sort. A large part of the history of mathematics was formed by various mathematicians trying to find the best way to represent the functions that resulted from the problems they were studying.

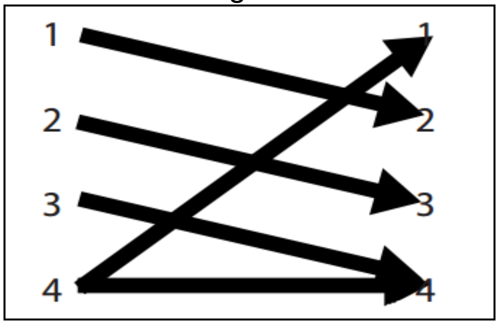
The blue and red road functions that came from a correctly colored city diagram are simple, but contain all the important parts of a function. No matter how complicated the functions we study this year get, we will constantly come back to the representations we developed in this early material: arrow diagrams, subgraphs, ordered pairs, and points plotted on a Cartesian coordinate plane.

What is important for us to recognize at this point is when an arrow diagram or ordered pairs represent a function (and when they do not!). It is also important to be able to find the domain and range of a function when you are given a representation of the function.

Let's do a few examples. For each of the representations below, we want to decide if it represents a function and, if it does, what the domain and the range are for the function.

When we write the domain or range of a function, we will write them as **sets**. A set in mathematics is just a collection of things put together. To symbolize this collection, we use set "braces". For example, if we wanted to form a set out of the numbers 1, 2, and 3 we would put the numbers between set braces like this, {1, 2, 3}. We use the term **element** to refer to the items inside a set.

For each of the following representations, first we will determine if it is a function, and if it is, we will write the domain and range as sets. Does this arrow diagram represent a function?



No! There are two arrows leaving from building number 4.

Figure 6.1a

Do the ordered pairs below represent a function? If we turn them into an arrow diagram the answer is easy to see.

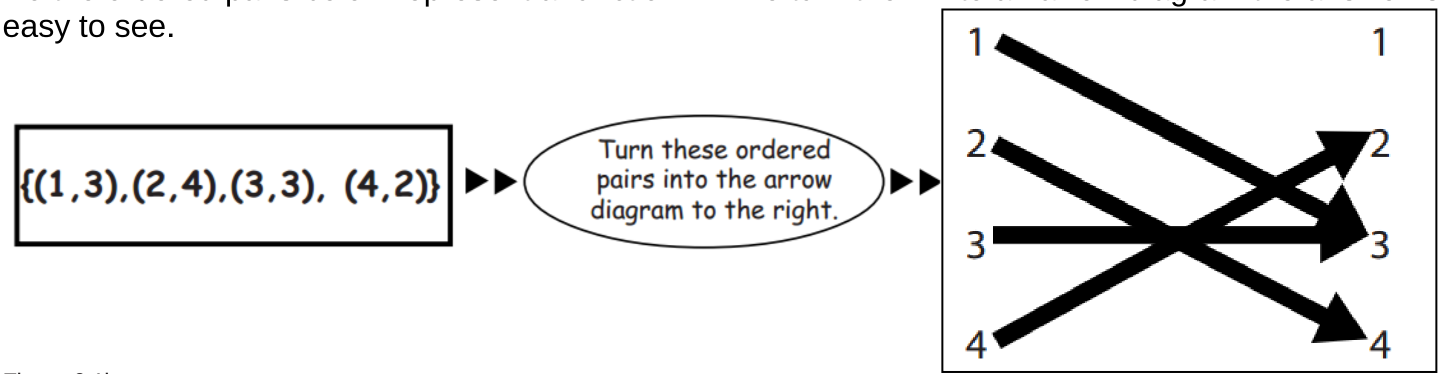


Figure 6.1b

Yes! There is **exactly one** arrow leaving from each building number. Since it is a function, let's list the domain and range as sets. Remember the domain is the set of "leave from" building numbers and the range is the set of "go to" building numbers.

The domain of the function is {1, 2, 3, 4} and the range is {2, 3, 4}. The number 1 is not in the range, since no arrow goes to building number 1 in the arrow diagram.

The box below contains the mathematical terms that are used for the elements in the domain and range of a function.

Table 6.1

Function Representation	Element that represents the DOMAIN	Element that represents the RANGE
Arrow diagram	"leave from" building number	"go to" building number
Road subgraph	"leave from" building circle	"go to" building circle
Ordered pair	1st coordinate	2nd coordinate
Graph of points	x-coordinate	y-coordinate

Team members: ____ *Team Name:* ________________

Activity 6.2 – TEAM WORK – Function Practice

For the ordered pairs below, draw the equivalent arrow diagram and determine if it is a function. If it is, write the domain and range of the function in the space provided.

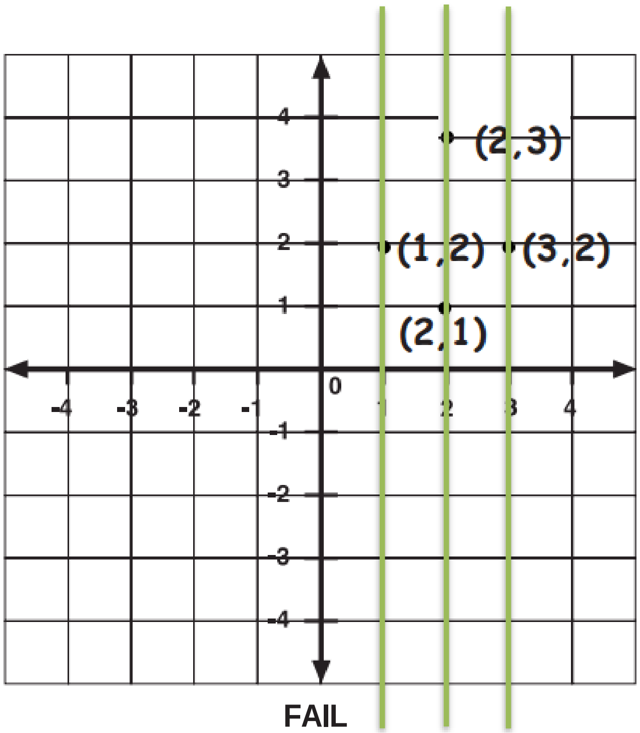
Table 6.2

Set of ordered pairs	Arrow Diagram	Is it a function?	Write the Domain and Range if it is a function
{ (1,2), (2,3), (3,4), (4,1) }	1 1 2 2 3 3 4 4	YES NO	Domain = { } Range = { }
{ (1,3), (2,3), (3,3), (4,3) }	1 1 2 2 3 3 4 4	YES NO	Domain = { } Range = { }
{ (1,3), (2,4), (3,4), (2,1), (4,3) }	1 1 2 2 3 3 4 4	YES NO	Domain = { } Range = { }
{ (1,2), (2,3), (3,1), (1,1), (4,2) }	1 1 2 2 3 3 4 4	YES NO	Domain = { } Range = { }
{ (1,1), (2,4), (3,1), (4,3) }	1 1 2 2 3 3 4 4	YES NO	Domain = { } Range = { }

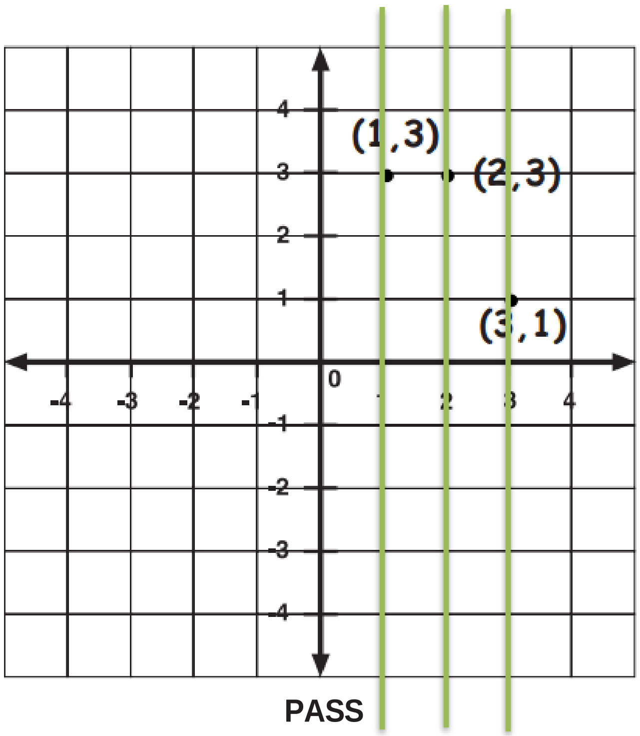
Activity 6.3 – CLASS WORK – Class Discussion: The Vertical Line Test

There are many ways to justify whether a relation is a function, but a simple one that you can use with a coordinate plane is called the Vertical Line Test. The Vertical Line Test is a quick check to see if something is not a function.

You perform the Vertical Line Test with ordered pairs by drawing (or imagining) a vertical line over every point. If a line hits more than one point, then the relation cannot be a function. Let's look at an example, with our lines in Green:



FAIL



PASS

Which points overlap above?

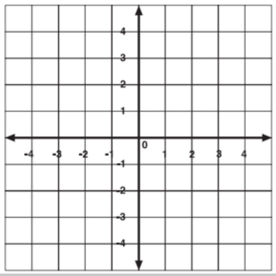
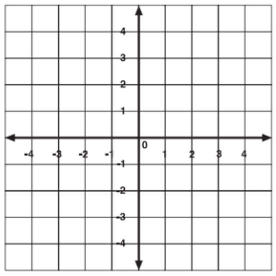
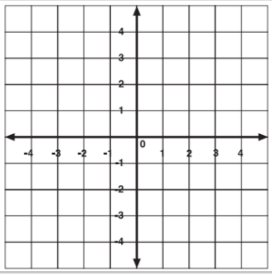
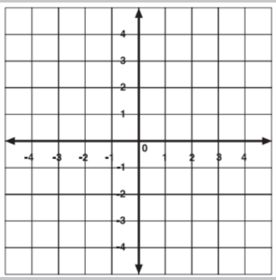
It's important to remember that the Vertical Line Test can't prove that a relation is a function, only that it is not.

Team members: ____ Team Name: ____________________

Activity 6.4 – TEAM WORK – Vertical Line Test Practice

For the ordered pairs below, draw the equivalent arrow diagram and determine if it is a function. If it is, write the domain and range of the function in the space provided.

Table 6.4

Set of ordered pairs	Plotted Points	Does it pass the Vertical Line Test?	Is it a function?
{ (1,2), (2,3), (3,4), (4,1) }		**YES** **NO**	**YES** **NO**
{ (1,3), (2,3), (3,3), (4,3) }		**YES** **NO**	**YES** **NO**
{ (1,3), (2,4), (3,4), (2,1), (4,3) }		**YES** **NO**	**YES** **NO**
{ (1,1), (2,4), (4,3) }		**YES** **NO**	**YES** **NO**

Does the Vertical Line Test tell you if something is a function? Why or why not?

Team members: ____ Team Name: __________________

Section 7 – Matrix Representations

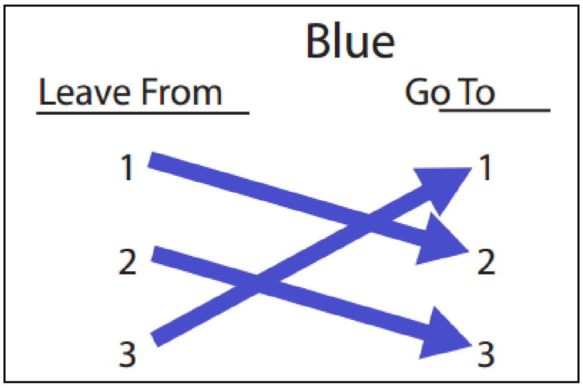
Activity 7.1 – TEAM WORK – A New Representation

You now have some experience with mathematical representations of the road functions. The last representation we will introduce is, in many ways, the most powerful. In fact, this last representation started a revolution. It has become one of the most widely studied objects in the history of mathematical history.

It is actually a very simple idea. In fact, with the materials you are provided you may be able to come up with the idea yourself.

Whatever representation you construct has to have enough information so that if you handed it to another student he or she would be able to figure out your original arrow diagram.

With the materials provided, find the last representation of the road functions. (Try to represent the arrow diagram below.)



Describe below why your team chose the representation you did. What features of the blue road arrow diagram does your team's representation capture?

Activity 7.2 – CLASS WORK – Class Discussion: Matrix Representations of Functions

Now we will show you how mathematicians construct this representation. We will use brackets (square parentheses) in our representation to give it a rectangular shape.

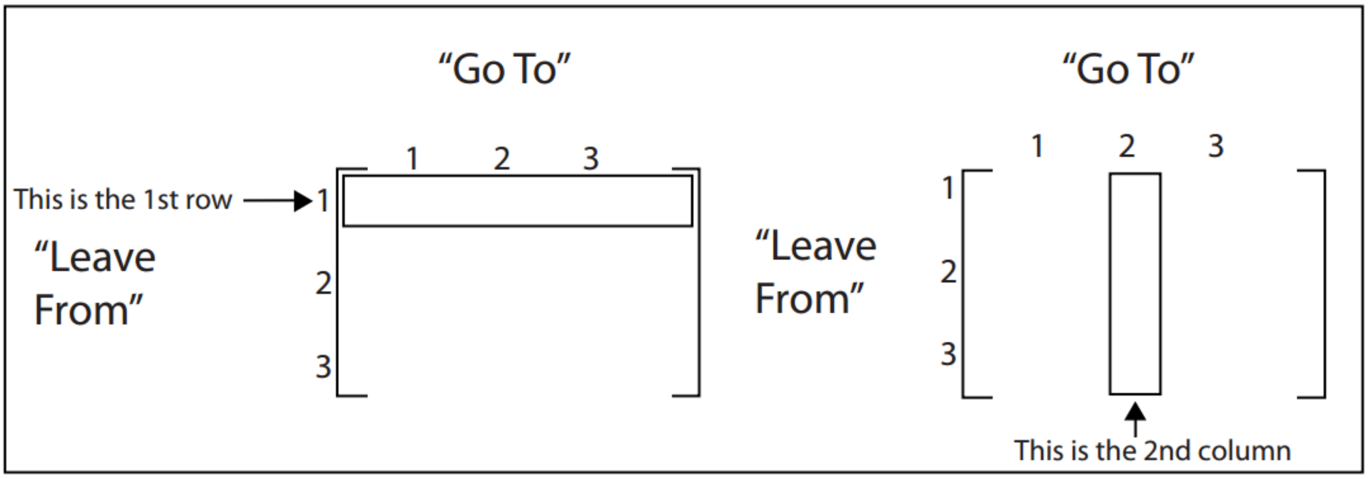
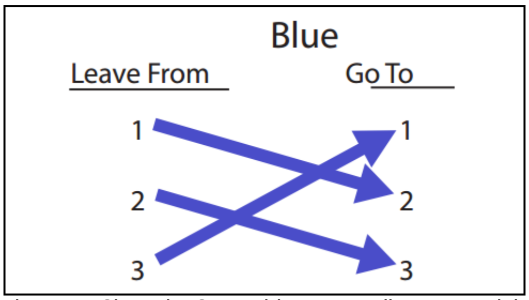


Figure 7.2a – Explanation of the rectangular grid representation of a function.

We designate positions in the rectangle by referring to the numbers. The numbers on the left refer to rows and the numbers on top refer to columns. In this way, each position in the rectangle has a row number and a column number associated with it. The row numbers will refer to "leave from" buildings and the column numbers will refer to "go to" buildings.

Here is the arrow diagram we were trying to represent. We'll also list the ordered pairs that we get from the arrow diagram. Remember that the ordered pair (1,2) means that there is a blue road that leaves from building 1 and goes to building 2.



Blue = {(1,2), (2,3), (3,1)}

Figures 7.2b and 7.2c – A blue arrow diagram and the set of blue ordered pairs that represent it.

We need a way to capture that same information in our rectangular grid. Just as you and your classmates might have done, we'll use different symbols to represent whether there is a road or not. A **1** will mean there is **one road** connecting two buildings and a **0** will mean there are **zero** roads connecting two buildings. So here is our rectangular grid of numbers. We'll put it next to the ordered pairs, so you can see the relationship between them.

Blue = {(1,2), (2,3), (3,1)}

$$\begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$

Figures 7.2d and 7.2e – A set of blue ordered pairs and the rectangular grid that represents it.

Below is another example of an arrow diagram and the ordered pairs and rectangular grid that is equivalent to it.

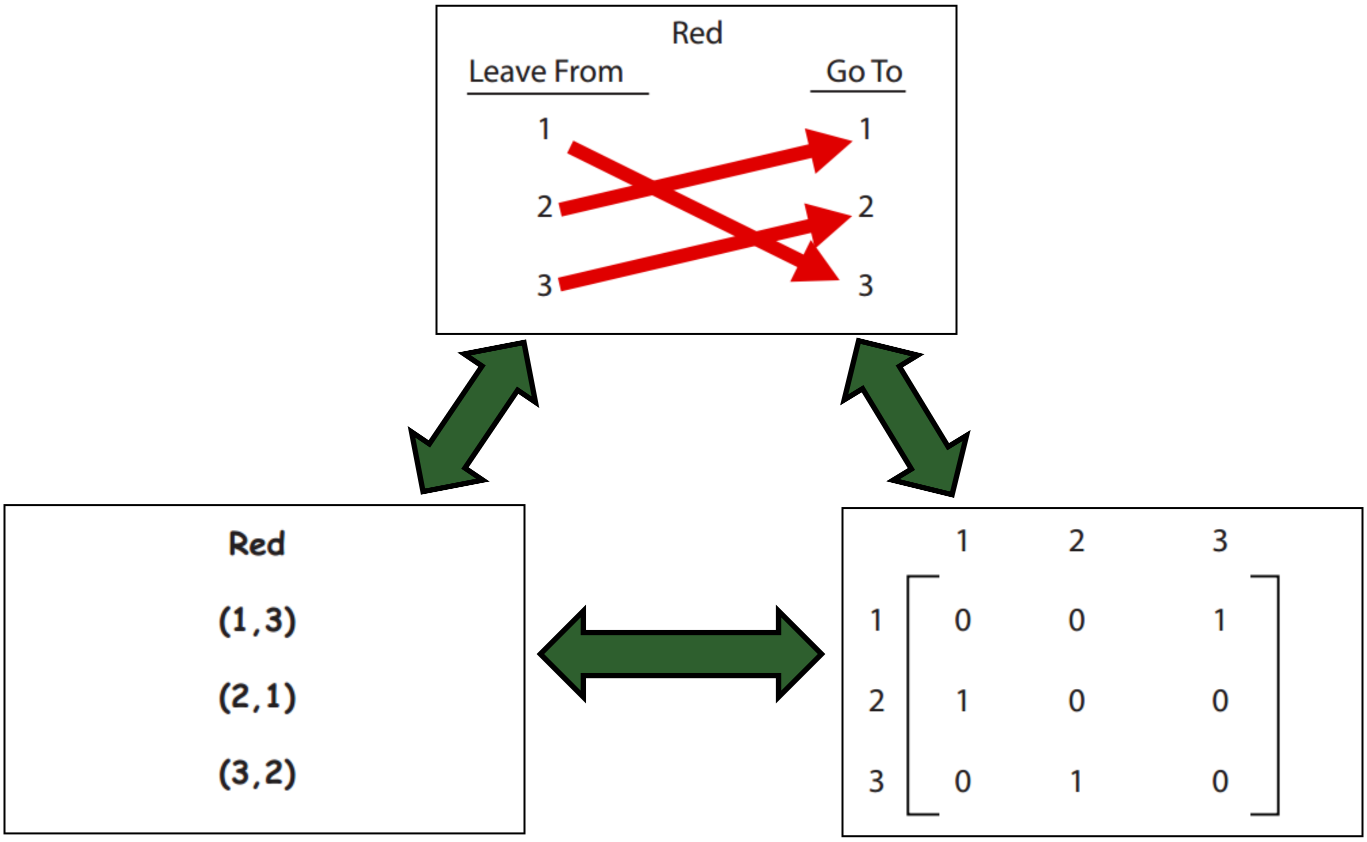


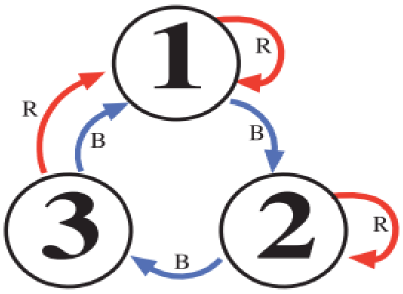
Figure 7.2f – A diagram showing an equivalent arrow diagram, set of ordered pairs, and rectangular grid, or "matrix", of a city.

We've been calling this representation a **rectangular grid of numbers** which is more of a description than a name. We will call this rectangular grid a **matrix** (the plural is **matrices**). The word matrix comes from the Latin word **mater** which means **mother.** It is because of this that you might say that a matrix is the "mother of all representations."

Team members: ____ Team Name: _________________

Activity 7.3 – TEAM WORK – Matrix Practice

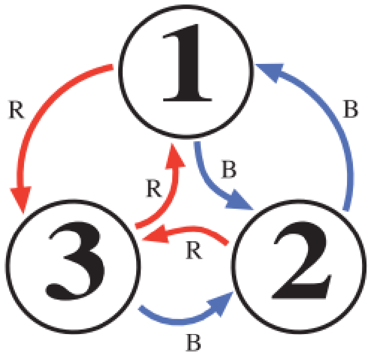
For each city below, draw the arrow diagrams and the associated matrix. Remember the row number provides the "leave from building numbers and the column number provides the "go to" building numbers.



Red	Go To
1 1 2 2 3 3	Leave From $\begin{array}{cc} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ \end{bmatrix} \end{array}$

Blue	Go To
1 1 2 2 3 3	Leave From $\begin{array}{cc} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ \end{bmatrix} \end{array}$

Team members: ____ Team Name: __________________



Red	
1	1
2	2
3	3

Go To

$$\text{Leave From}\quad \begin{array}{cc} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} \underline{\qquad} & \underline{\qquad} & \underline{\qquad} \\ \underline{\qquad} & \underline{\qquad} & \underline{\qquad} \\ \underline{\qquad} & \underline{\qquad} & \underline{\qquad} \end{bmatrix} \end{array}$$

Blue	
1	1
2	2
3	3

Go To

$$\text{Leave From}\quad \begin{array}{cc} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} \underline{\qquad} & \underline{\qquad} & \underline{\qquad} \\ \underline{\qquad} & \underline{\qquad} & \underline{\qquad} \\ \underline{\qquad} & \underline{\qquad} & \underline{\qquad} \end{bmatrix} \end{array}$$

Section 8 – A New Operation

Activity 8.1 – CLASS WORK – Class Discussion: Function Composition

We are all familiar with the operations of arithmetic: addition, subtraction, multiplication, and division. In mathematics, every time a new type of object is created we immediately wonder if there is a new operation for the objects. Let's try to discover an operation for the arrow diagrams. What we will do is find a way to combine two arrow diagrams together to form a new one. We'll use the city below as our example.

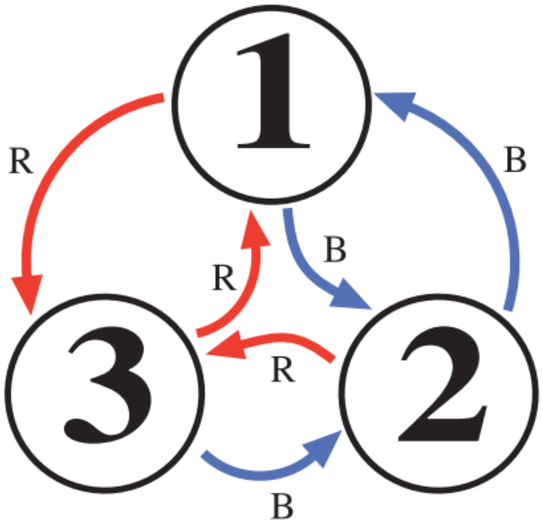


Figure 8.1a

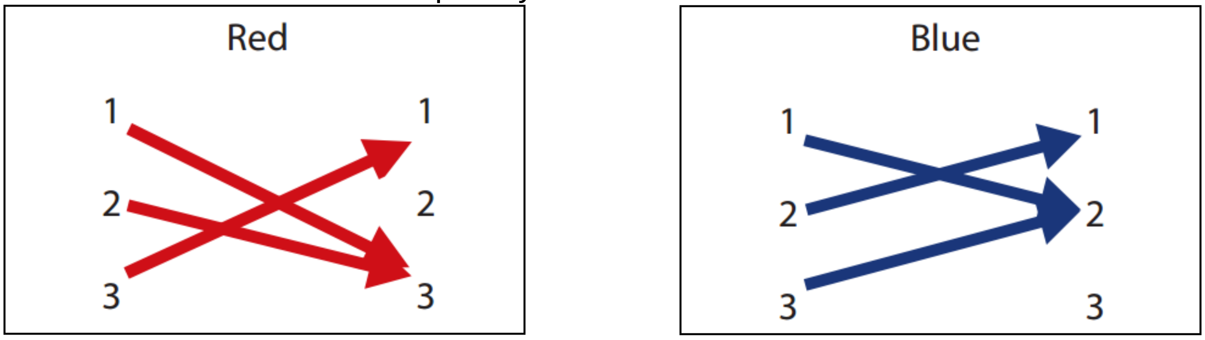
We need three students to walk the city and one student to call out two instructions. Keep track of the students as they walk through the city, and represent the combination of the two instructions as one arrow diagram. Try two different combinations (For example: blue-blue and red-blue).

In the boxes below try to represent the combined effect of the two instructions called as one arrow diagram. Make sure that you watch carefully as the students walking the city follow the two instructions called.

Table 8.1

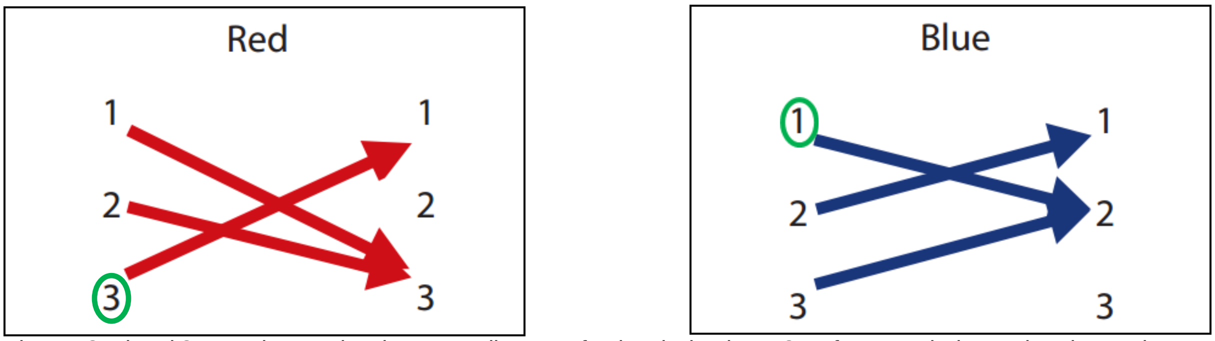
Arrow diagram for first instruction called		Arrow diagram for second instruction called		Combined arrow diagram of the 2 instructions called	
1	1	1	1	1	1
2	2	2	2	2	2
3	3	3	3	3	3
1	1	1	1	1	1
2	2	2	2	2	2
3	3	3	3	3	3

Now we'll describe how mathematicians combine arrow diagrams. Below are the arrow diagrams of the red and blue road functions for the example city.



Figures 8.1b and 8.1c – Arrow diagrams for the city in Figure 8.1a

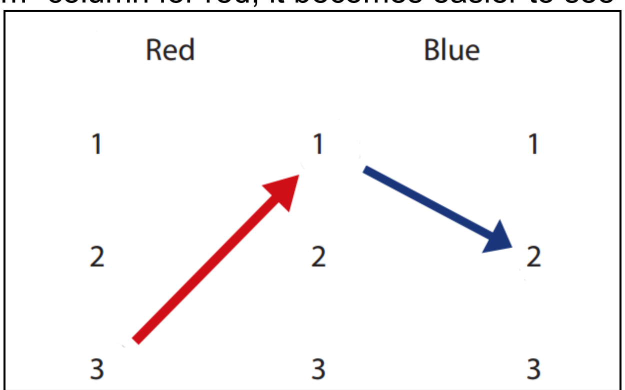
Suppose the two instructions we want to combine are "red-blue". We'll put the arrow diagrams for blue and red next to each other. We will concentrate on the person who starts in building 3. Suppose Candice starts there.



Figures 8.1d and 8.1e – Figures showing arrow diagrams for the city in Figure 8.1a for a "Red" then "Blue" instruction.

When "red" is called first, Candice goes to building 1. Building 3 is the "leave from" building and building 1 is the "go to" building. But building 1 becomes the "leave from" building for Candice when "blue" is called and Candice now follows the blue road to go to building 2. If we make the "go to" column for blue the same as the "leave from" column for red, it becomes easier to see what happens to Candice.

Figure 8.1f



So the combined effect is:

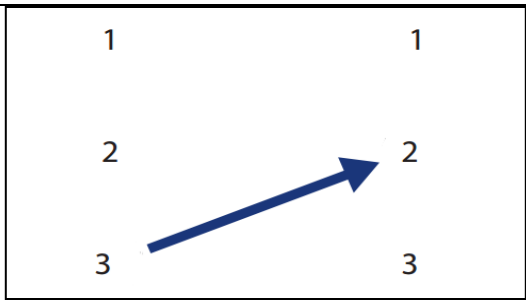
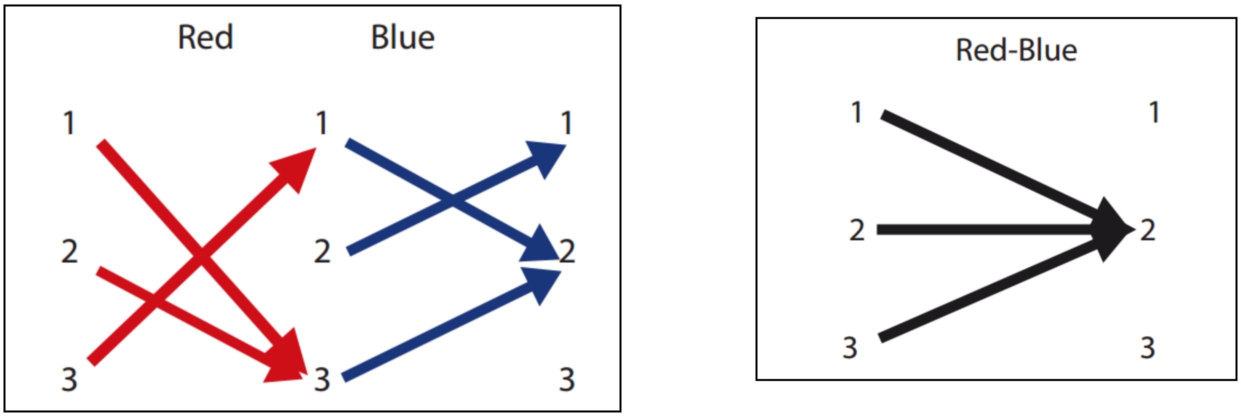


Figure 8.1g

Now we'll put all the arrows in our two arrow diagrams and find the combined effect for each person in the city.



Figures 8.1h and 8.1i – Figures showing the combined and complete arrow diagrams for the city in Figure 8.1a with a "red then "blue" call.

That's right. The instruction "red-blue" is a synchronizing instruction!! The combined action of "red-blue" gets everyone to building 2. Is this a synchronizing instruction that you found earlier?

This binary operation which combines two arrow diagrams into one new arrow diagram is called **composition**. We need a special symbol for representing the operation. We will start to use the letter B to represent the instruction "blue" and R to represent the instruction "red". The symbol for composition will be "∘".

In other words, if someone calls out "red-blue" and we want to combine the arrow diagrams for red and blue into a single arrow diagram we will write R ∘ B. Here is how we will represent composing the commands using our new symbols.

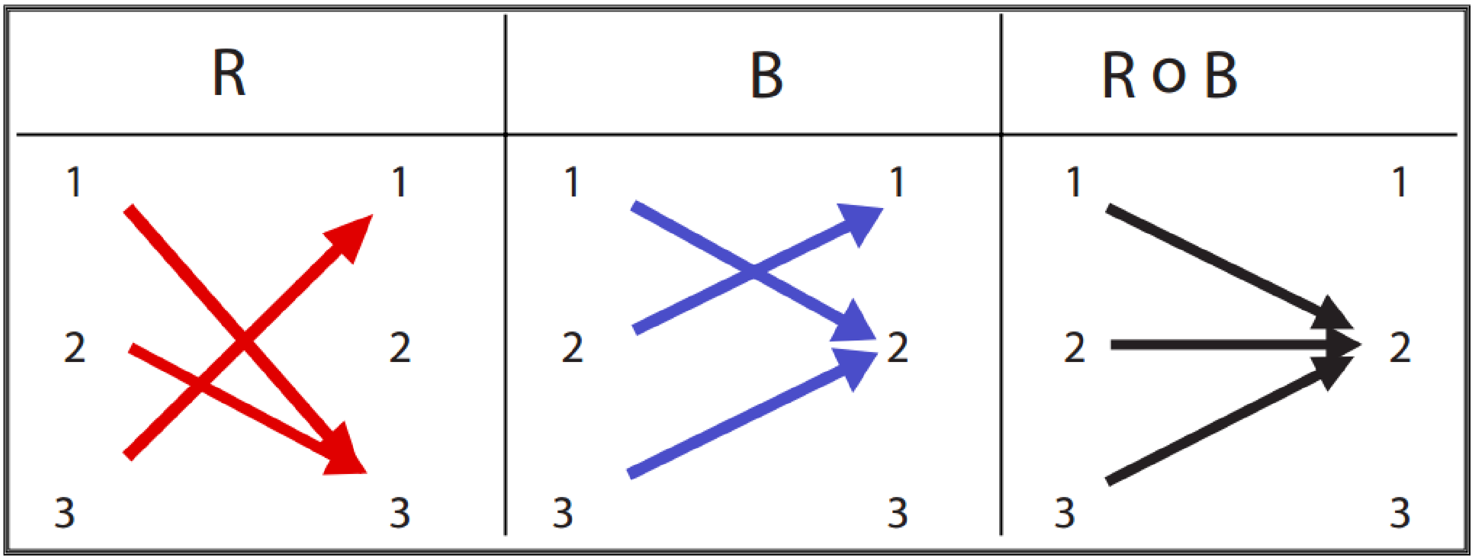
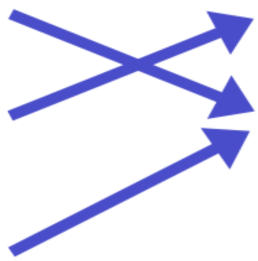
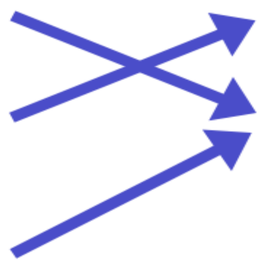


Figure 8.1j – The individual, then composed arrow diagrams for the city in Figure 8.1a.

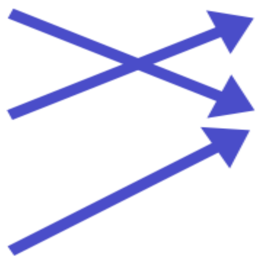
Team members: ____ Team Name: ____________________

Activity 8.2 – TEAM WORK – Combining Arrow Diagrams

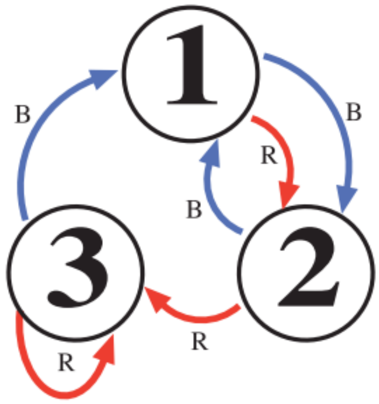
Find the combined arrow diagram for each of the other possible instructions (blue-blue, red-red, blue-red), then complete the arrow diagrams from the directed graphs.

B	B	B o B
1 1 2 2 3 3 	1 1 2 2 3 3 	1 1 2 2 3 3

R	R	R o R
1 1 2 2 3 3 	1 1 2 2 3 3 	1 1 2 2 3 3

B	R	B o R
1 1 2 2 3 3 	1 1 2 2 3 3 	1 1 2 2 3 3

Team members: ____ Team Name: _________________



B			R	
1	1		1	1
2	2		2	2
3	3		3	3

B followed by B

1	1	1
2	2	2
3	3	3

B o B

1	1
2	2
3	3

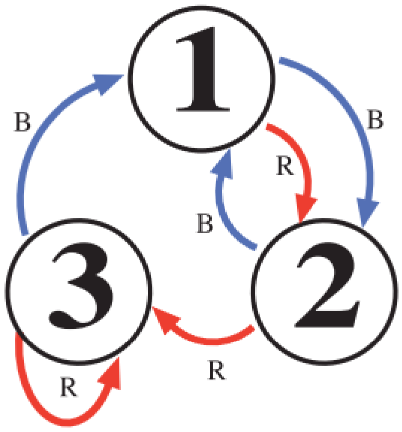
B followed by R

1	1	1
2	2	2
3	3	3

B o R

1	1
2	2
3	3

Team members: ____ *Team Name:* _________________



R followed by B

1	1	1
2	2	2
3	3	3

$R \circ B$

1	1
2	2
3	3

R followed by R

1	1	1
2	2	2
3	3	3

$R \circ R$

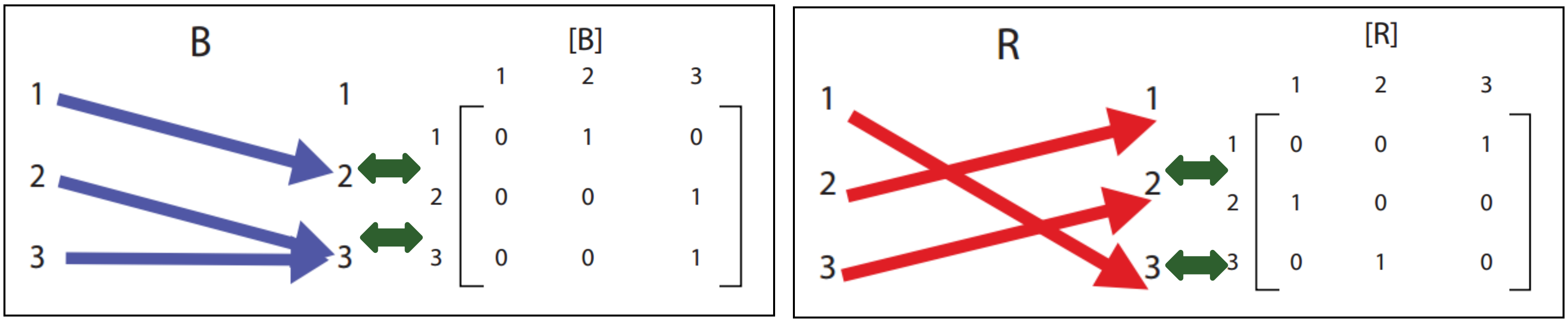
1	1
2	2
3	3

Activity 8.3 – CLASS WORK – Class Discussion: Multiplying Road Matrices

In our last activity, we defined a new operation for the arrow diagrams. In this activity we will take advantage of the relationship between the arrow diagrams and the road matrices to find an equivalent operation for the matrices.

What really happens when we compose the arrow diagrams together? Suppose we were given the following two arrow diagrams. To save space we will stop writing “leave from” and “go to” above their columns in the arrow diagrams. By now you should know that the arrow always leaves from the “leave from” building and goes to the “go to” building.

It really doesn’t matter if we use the arrow diagrams or the road matrices to represent what is happening. We could just as easily write the matrices for the arrow diagrams. Whenever we write the matrix for a road arrow diagram we will put brackets around the name so that you know that it is a matrix. In other words the matrix for the arrow diagram B will be [B], (we read this as “matrix B”). Also the matrix for the arrow diagram R will be [R] (read as “matrix R”).

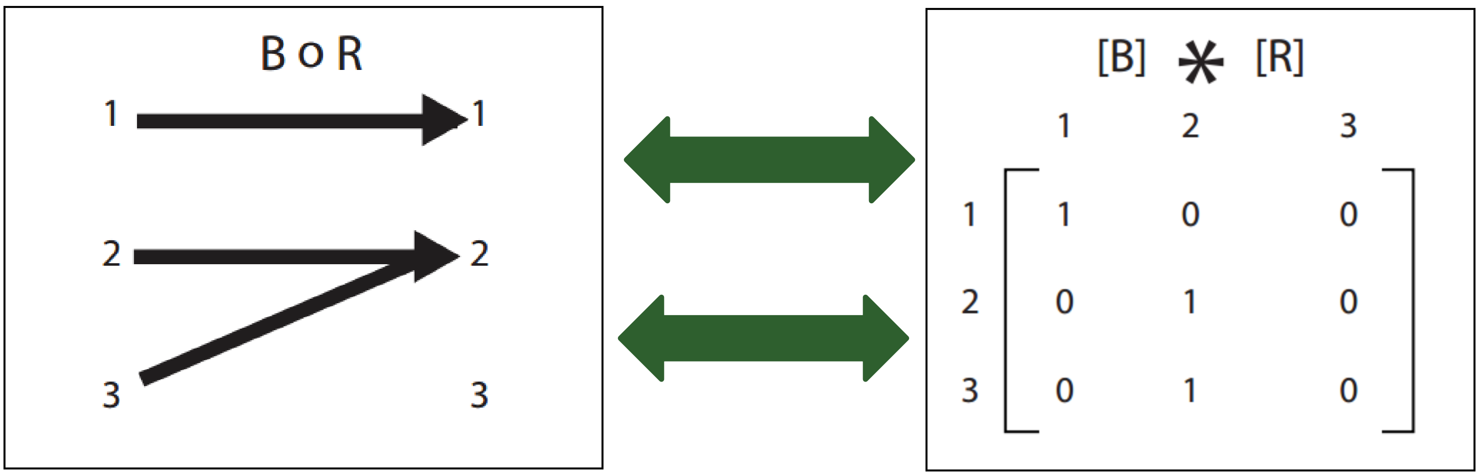


$$[B] = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \\ 3 & 0 & 0 & 1 \end{array} \qquad [R] = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0 & 0 & 1 \\ 2 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \end{array}$$

Figures 8.3a and 8.3b – Arrow diagrams with their matching matric representations.

Since we will be combining matrices together we will need a symbol for this operation. We will use “*”. This new operation for matrices is equivalent to composition of the arrow diagrams. We will refer to this operation as multiplication of the matrices.

The composition of the blue and red arrow diagrams is below, and to its right is the product of the two matrices [B] and [R]. We will define [B] * [R] to be the same matrix as [B ∘ R].



$$[B] * [R] = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 3 & 0 & 1 & 0 \end{array}$$

Figure 8.3c and 8.3d – The arrow diagram composition and matrix multiplication representations of function composition.

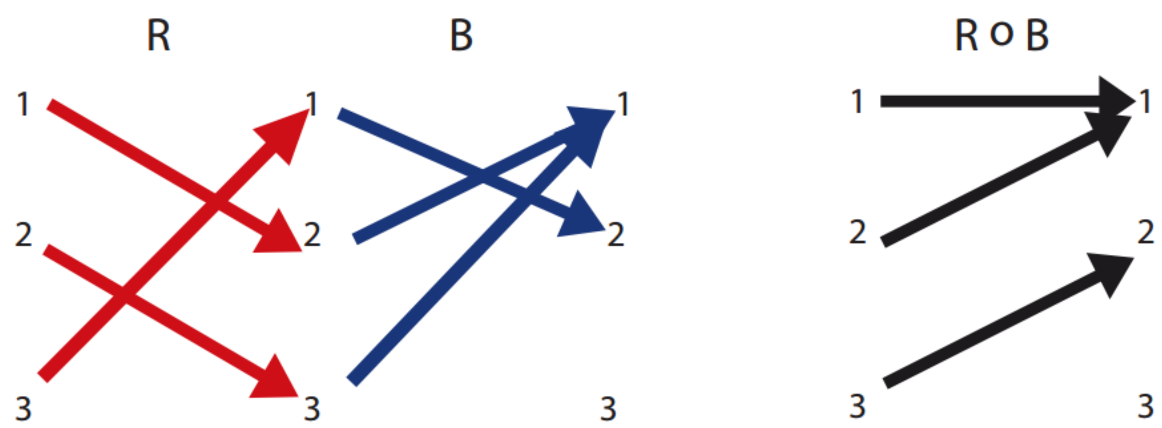
The easiest way to multiply road matrices is to convert them back into arrow diagrams, compose the arrow diagrams, and then convert the combined arrow diagram back into a matrix. In other words, if we start with these matrices.

$$[R] \qquad\qquad [B]$$

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \end{array} \qquad \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$

Figure 8.3e – Matrix representations for R and B.

Then to find [R]∘[B], we convert them into arrow diagrams and compose the arrow diagrams together.



$$[R] * [B]$$

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

Figure 8.3f – The process of matrix multiplication via arrow diagrams.

Composing the arrow diagrams together is easy, since all you have to do is follow the arrows. Matrix multiplication is a little trickier, but let's see if we can begin to understand it better. To do this, we will concentrate on the movement of one person in our city. Let's look at Candice who starts in building 1, and follows the instruction "red-blue".

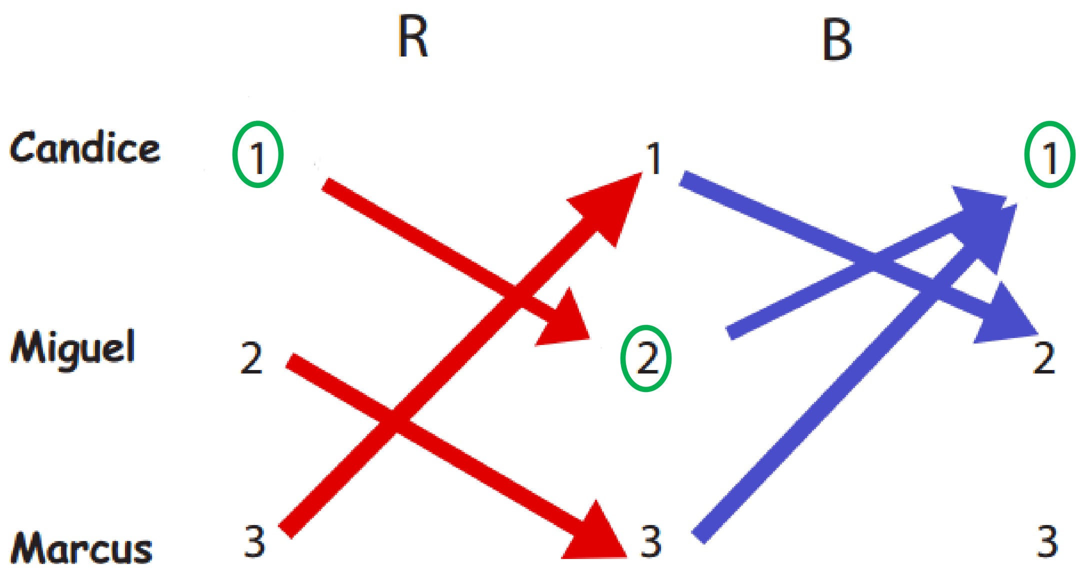


Figure 8.3g – Following Candice through 2 instructions.

Candice leaves from building 1 and goes to building 2 using the red road. Therefore, we have a 1 in the 1st row, 2nd column position of [R]. Next Candice leaves from building 2 and goes to building 1 using the blue road, so there is a 1 in the 2nd row, 1st column position of [B].

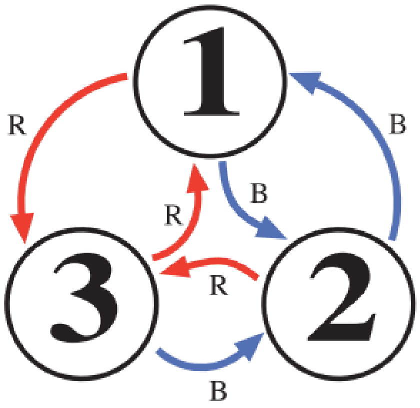
$$[R] \quad \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \end{array} \qquad [B] \quad \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$

Figure 8.3h – Following Candice through 2 instructions in matrix form.

Team members: ____ Team Name: __________________

Activity 8.4 – TEAM WORK – Matrix Multiplication

For the city below, draw the arrow diagram for the road functions, find the associated matrices, then multiply the given matrices together by combining the associated arrow diagrams.

B			R	
1	1		1	1
2	2		2	2
3	3		3	3

$$[B] = \begin{array}{cc} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ \end{bmatrix} \end{array} \qquad [R] = \begin{array}{cc} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ \end{bmatrix} \end{array}$$

B o R	
1	1
2	2
3	3

$$[B] * [R] = \begin{array}{cc} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ \end{bmatrix} \end{array}$$

Name: ______________________________ Teacher: ____________________________ Date: _____________

Activity 8.5a – INDIVIDUAL WORK – Function Composition and Matrix Multiplication

Complete the table below by composing the arrow diagrams.

Composition ○	$1 \to 1$ $2 \to 2$ $3 \to 3$	$1 \to 3$ $2 \to 1$ $3 \to 2$	$1 \to 2$ $2 \to 3$ $3 \to 1$	$1 \to 1$ $2 \to 3$ $3 \to 2$	$1 \to 3$ $2 \to 2$ $3 \to 1$	$1 \to 2$ $2 \to 1$ $3 \to 3$
$1 \to 1$ $2 \to 2$ $3 \to 3$	1 1 2 2 3 3	1 1 2 2 3 3	1 1 2 2 3 3	1 1 2 2 3 3	1 1 2 2 3 3	1 1 2 2 3 3
$1 \to 3$ $2 \to 1$ $3 \to 2$	1 1 2 2 3 3	1 1 2 2 3 3	1 1 2 2 3 3	1 1 2 2 3 3	1 1 2 2 3 3	1 1 2 2 3 3
$1 \to 2$ $2 \to 3$ $3 \to 1$	1 1 2 2 3 3	1 1 2 2 3 3	1 1 2 2 3 3	1 1 2 2 3 3	1 1 2 2 3 3	1 1 2 2 3 3
$1 \to 1$ $2 \to 3$ $3 \to 2$	1 1 2 2 3 3	1 1 2 2 3 3	1 1 2 2 3 3	1 1 2 2 3 3	1 1 2 2 3 3	1 1 2 2 3 3
$1 \to 3$ $2 \to 2$ $3 \to 1$	1 1 2 2 3 3	1 1 2 2 3 3	1 1 2 2 3 3	1 1 2 2 3 3	1 1 2 2 3 3	1 1 2 2 3 3
$1 \to 2$ $2 \to 1$ $3 \to 3$	1 1 2 2 3 3	1 1 2 2 3 3	1 1 2 2 3 3	1 1 2 2 3 3	1 1 2 2 3 3	1 1 2 2 3 3

Name: ______________________________ Teacher: ____________________________ Date: _____________

Activity 8.5b – INDIVIDUAL WORK – Function Composition and Matrix Multiplication

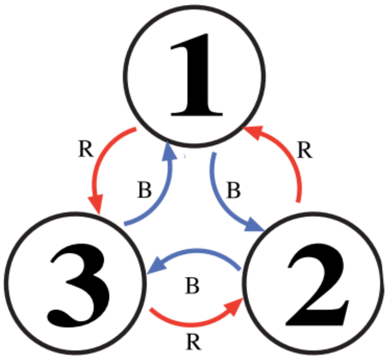
Complete the table below by using the arrow diagrams in Activity 9.5a to multiply the matrices.

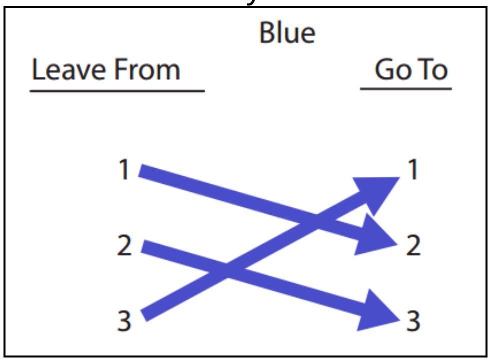
Matrix Multiplication *	$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$	$\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}$	$\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}$	$\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}$	$\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}$	$\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}$
$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$						
$\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}$						
$\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}$						
$\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}$						
$\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}$						
$\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}$						

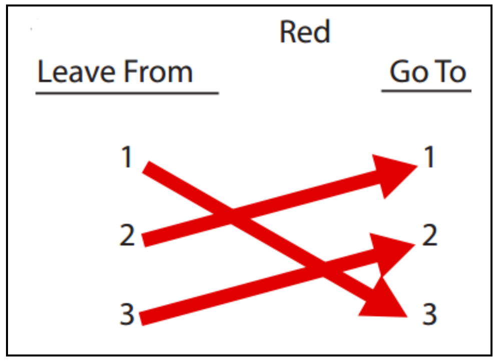
Section 9 – Function Notation

Activity 9.1 – CLASS WORK – Class Discussion: Function Notation

As we have seen in our discussion about functions, there are many different ways to represent or symbolize the information in a function. There is one more important piece of notation for a function that emphasizes the "input-output machine". Here is a city with its arrow diagrams:







How can we emphasize the "input-output" nature of the function?

Let's look at single actions from each arrow diagram, with both students starting at 1:

$1 \xrightarrow{\text{Blue}} 2$

- The student moves from 1 to 2 when "BLUE" is called, as in the arrow diagram. It reads "start at 1, then walk to 2 when "BLUE" is called".

$1\ \text{Blue} = 2$

- We can replace the arrow with an equals sign. Now it reads "starting at 1 and calling "BLUE" is the same as 2".

$1\ B = 2$

- Replacing the word "Blue" with just "B" shortens it further, so now it reads "starting at 1 and calling "B" is the same as 2".

$(1)B = 2$

- We can take out the circles to make it a little easier to write.

$B(1) = 2$

- Finally, we switch around the input and the function name to give us **function notation**. It is read "B of 1 equals 2".

Another example with Red:

$1 \xrightarrow{\text{Red}} 3$

- The student moves from 1 to 3 when "RED" is called, as in the arrow diagram. It reads "start at 1, then walk to 3 when "RED" is called".

$1\ \text{Red} = 3$

- We can replace the arrow with an equals sign. Now it reads "starting at 1 and calling "RED" is the same as 3".

$1\ R = 3$

- Replacing the word "Red" with just "R" shortens it further, so now it reads "starting at 1 and calling "R" is the same as 3".

$(1)R = 3$

- We can take out the circles to make it a little easier to write.

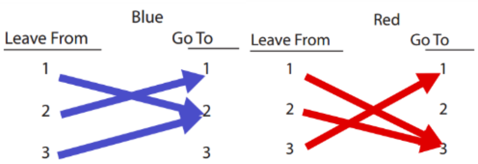
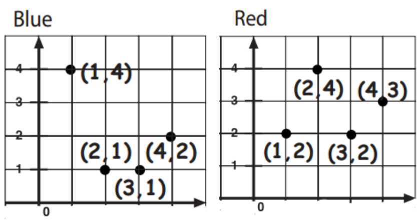
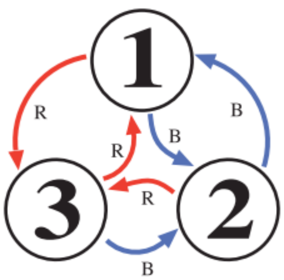
$R(1) = 3$

- Finally, we switch around the input and the function name to give us **function notation**. It is read "R of 1 equals 3".

Team members: ____ Team Name: ________________

Activity 9.2 – TEAM WORK – Function Notation Practice

Fill in the function notation for each city representation below.

City Representation	Blue Function Notations	Red Function Notations
		
		
Blue {(1,3), (2,4), (3,3), (4,1)} **Red** {(1, 1), (2,1), (3,2), (4,3)}		
[R] $\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$ [B] $\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$		
		

Activity 9.3 – CLASS WORK – Class Discussion: Composing Function Notations

Just like how our city's arrow diagrams could be **composed**, we can compose function notation as well. Let's take another look at an example of a function composition from last section:

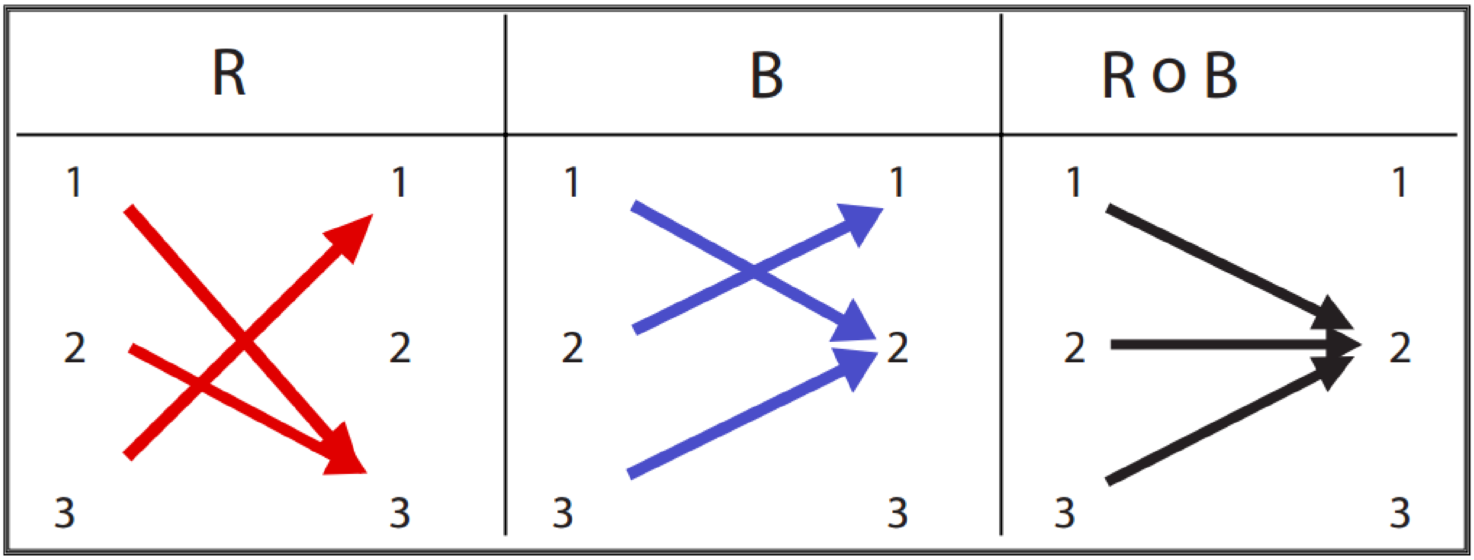


Figure 9.3a – Arrow diagrams depicting function composition.

Let's use our new notation to describe what happens when we start at "1". R(1) = 3 because we would travel from Building 1 to Building 3 when RED is called. Then, when BLUE is called, we'll travel from location 3 to location 2, which looks like B(3) = 2 in our new function notation.

What about a shortcut for this notation? When we composed the arrow diagrams we used R∘B as a shortcut, but that doesn't fit our current function notation. The answer is that we can compose functions by putting one inside another. Here's the example rewritten:

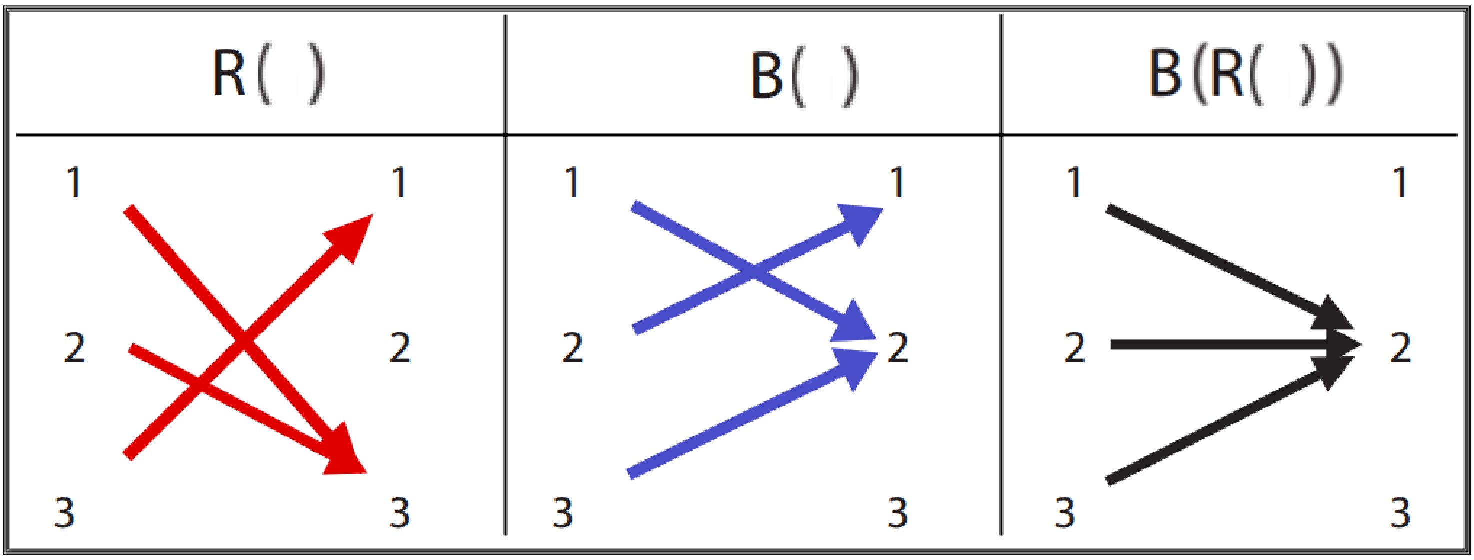


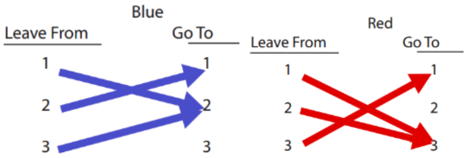
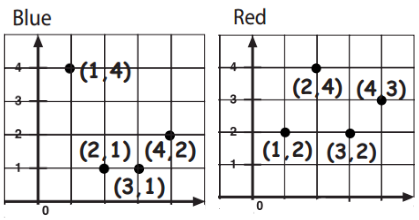
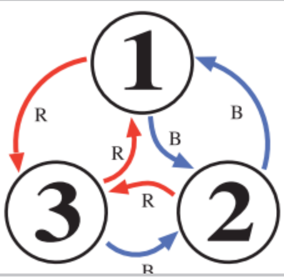
Figure 9.3b – Arrow diagrams depicting function composition with function notation.

The above example of doing R(1) = 3 and then B(3) = 2 becomes B(R(1)). This makes sense if you remember your rules about doing what comes in parentheses first. We already know that R(1) = 3, so if we replace R(1) with 3 we'd get B(3). We know that B(3) = 2, so it makes sense that B(R(1)) = 2.

Team members: ____ Team Name: ________________

Activity 9.4 – TEAM WORK – Function Notation Composition Practice

Complete the table below with all of the values from the function compositions.

City Representation	Blue Function Notations Red Function Notations	
	B(R(1)) =	R(B(1)) =
	B(R(2)) =	R(B(2)) =
	B(R(3)) =	R(B(3)) =
	B(R(1)) =	R(B(1)) =
	B(R(2)) =	R(B(2)) =
	B(R(3)) =	R(B(3)) =
Blue {(1,3), (2,4), (3,3), (4,1)} **Red** {(1, 1), (2,1), (3,2), (4,3)}	B(R(1)) =	R(B(1)) =
	B(R(2)) =	R(B(2)) =
	B(R(3)) =	R(B(3)) =
[R] $\begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$ [B] $\begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$	B(R(1)) =	R(B(1)) =
	B(R(2)) =	R(B(2)) =
	B(R(3)) =	R(B(3)) =
	B(R(1)) =	R(B(1)) =
	B(R(2)) =	R(B(2)) =
	B(R(3)) =	R(B(3)) =

Section 10 – Inverses

Activity 10.1 – CLASS WORK – Class Discussion: Inverses

We have discussed many models and representations of cities. Some cities could be defined as "functions," and others could not. In this section, we will discuss a new change we can explore called an "**inverse**" of a function.

The inverse of an action is the action that reverses it to the state that it was in before. Let's look at inverses in our cities:

- If we call "Red", what does everyone do? Write in their names in the correct spot in Table 10.1a.
  - Candace –
  - Miguel –
  - Marcus –
- If we call "undo Red", what should everyone do? Using the city you drew in Table 10.1a, write in arrows and their names in the correct spot in Table 10.1b. What did they do?
  - Candace –
  - Miguel –
  - Marcus –
- If we call "Blue", what does everyone do? Write in their names in the correct spot in Table 10.1c.
  - Candace –
  - Miguel –
  - Marcus –
- If we call "undo Blue", what should everyone do? Using the city you drew in Table 10.1c, write in their names in the correct spot in Table 10.1d. What did they do?
  - Candace –
  - Miguel –
  - Marcus –

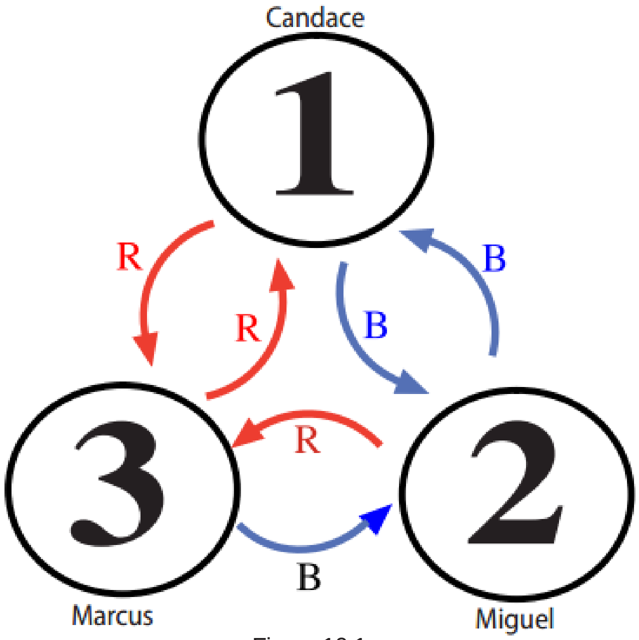
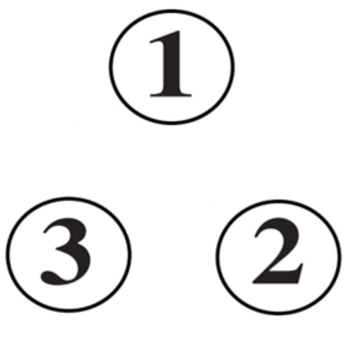
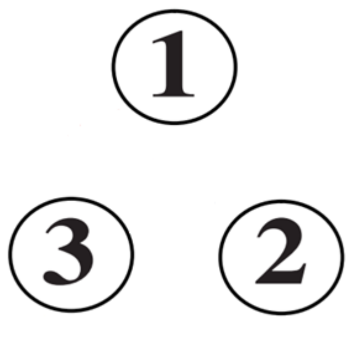
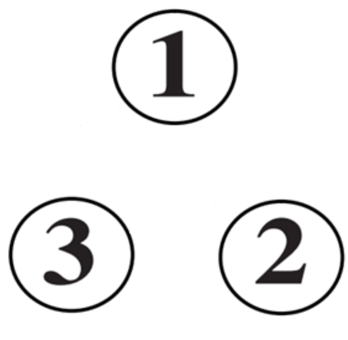
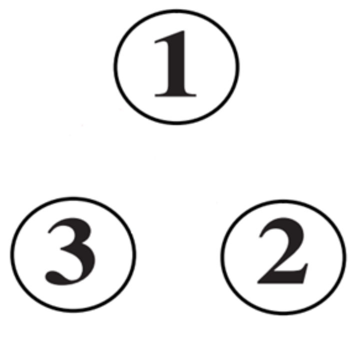


Figure 10.1

Table 10.1

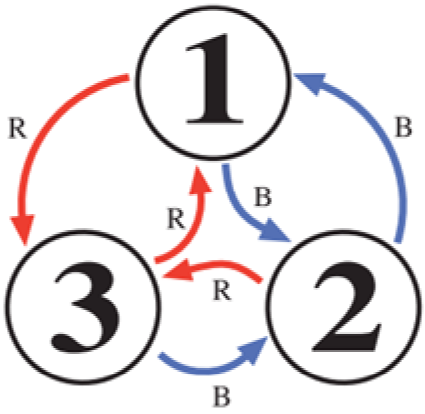
A. After calling "Red"	**B. After calling "undo Red"**	**C. After calling "Blue"**	**D. After calling "undo Blue"**
			

Box B is the inverse of box A, and box D is the inverse of box C.

Team members: ____ *Team Name:* _________________

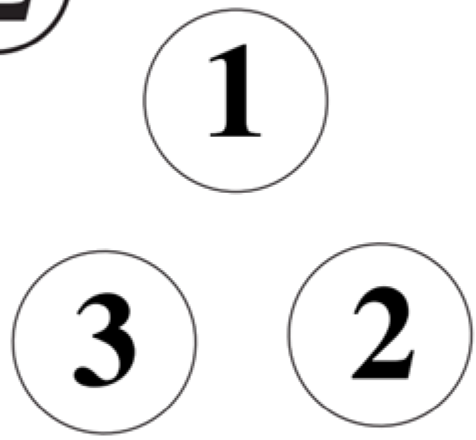
Activity 10.2 – TEAM WORK – Inverse Practice with Multiple Representations

For the cities below, draw the arrow diagram and graph the ordered pairs for the **inverses** of the functions of the blue and red roads.



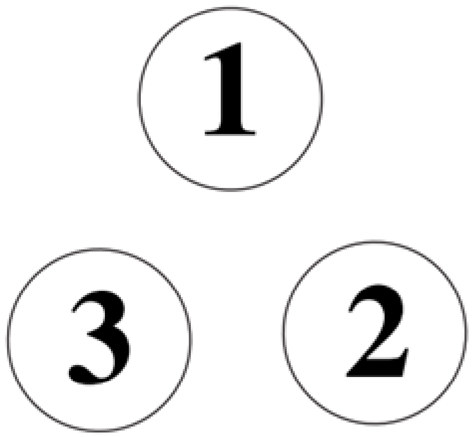
Inverse of Red

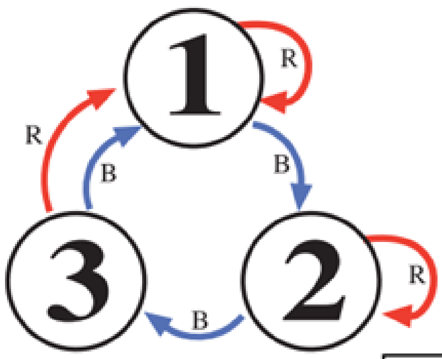
Leave From	Go To
1	1
2	2
3	3



Inverse of Blue

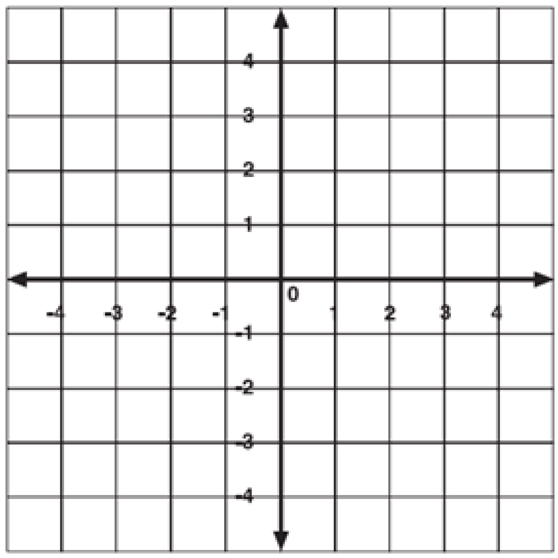
Leave From	Go To
1	1
2	2
3	3

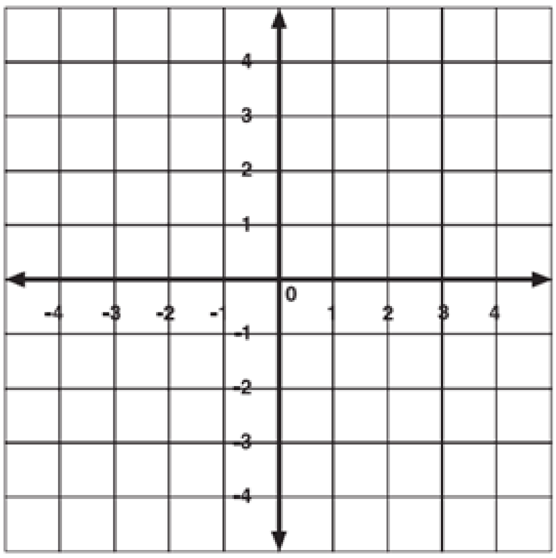




Inverse of Blue		Ordered pairs for B
Leave From	Go To	
1	1	
2	2	
3	3	

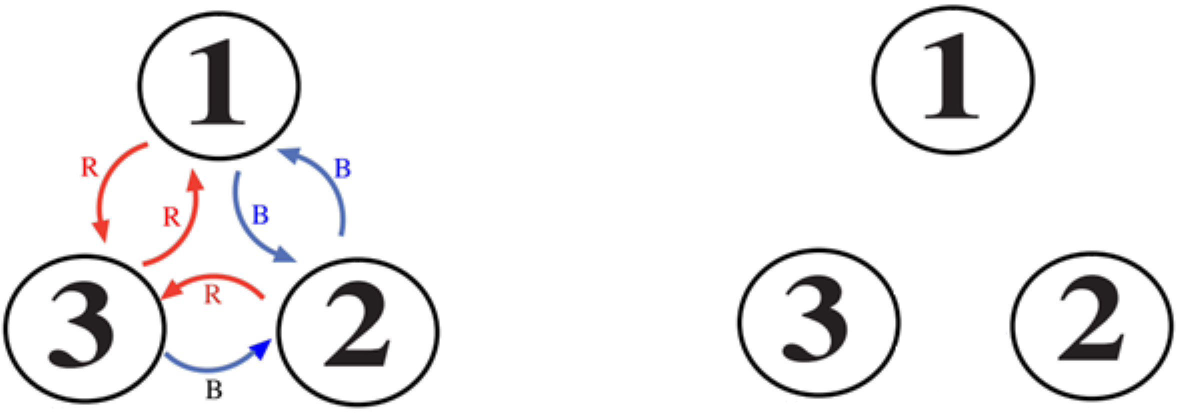
Inverse of Red		Ordered pairs for R
Leave From	Go To	
1	1	
2	2	
3	3	





Activity 10.3 – CLASS WORK – Class Discussion: Domains and Ranges of Inverses and if they are Inverse Functions

Inverses of functions have domains and ranges just like functions do. Let’s look at the domains and ranges of the city we looked at in Activity 10.1. Start by drawing the inverse of the city in Activity 10.1.



Figures 10.3a and 10.3b – The city in Figure 10.1 and its inverse

Domain of 10.3a	**Range of 10.3a**	**Domain of 10.3b**	**Range of 10.3b**

What happened to the domain and range when you found the inverse?

Now let’s look at them as functions. It may be easier to start by turning them into arrow diagrams first.

Arrow Diagram of Red in 10.3a	**Arrow Diagram of Blue in 10.3a**	**Arrow Diagram of Red in 10.3b**	**Arrow Diagram of Blue in 10.3b**
1 1	1 1	1 1	1 1
2 2	2 2	2 2	2 2
3 3	3 3	3 3	3 3

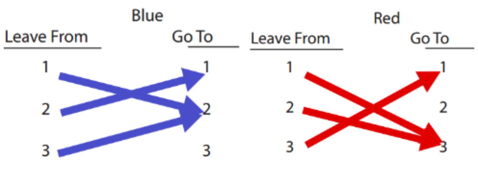
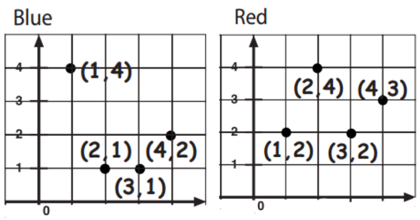
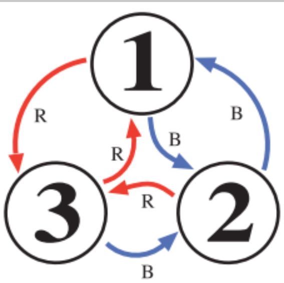
What happened to the arrow diagrams when you found the inverse?

Are the inverses functions? Why or why not?

Team members: ____ Team Name: _________________

Activity 10.4 – TEAM WORK – Inverse Function Practice

Fill in the inverse of each representation below and answer whether the inverse is a function.

City Representation	Inverse of Red	Inverse of Blue
	Is it a function? Why?	Is it a function? Why?
	Is it a function? Why?	Is it a function? Why?
Blue {(1,3), (2,4), (3,3), (4,1)} **Red** {(1, 1), (2,1), (3,2), (4,3)}	Is it a function? Why?	Is it a function? Why?
[R] $\begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$ [B] $\begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$	Is it a function? Why?	Is it a function? Why?
	Is it a function? Why?	Is it a function? Why?

Glossary of Terms

Section #	Term	Example	Mathematical Representation
	Directed Graph		
	Vertex/Vertices		
	Directed Edges		
	Arrow Diagram		
	Subgraph		
	Ordered Pair		
	First Coordinate, Second Coordinate		
	Axis		
	Coordinate Axes		
	Horizontal		
	Vertical		

	Cartesian Coordinate System		
	Plane		
	x-coordinate		
	y-coordinate		
	Function		
	Domain		
	Range		
	Set		
	Element		
	Composition		
	Matrix/Matrices		
	Inverse		